1 Answers to Chapter 3, Odd-numbered Exercises

1) $r(n) = 25r(n-1) + 3r(n-2) + 10^{n-1}$. There are $25r(n-1)$ identifiers satisfying the first condition, $3r(n-2)$ satisfying the second condition, and 10^{n-1} satisfying the third condition. $r(5) = 10605609$.

3) $g(n) = 2g(n-1) + (g(n-1) - g(n-3)) = 3g(n-1) - g(n-3)$. $g(1) = 3$, $g(2) = 9$, $g(3) = 26$. Our recursion involves looking back 3 terms ($g(n-3)$), so we need to specify 3 initial values. Ternary strings are strings over the alphabet $\{0, 1, 2\}$. We can form a valid 102-avoiding string of length n as follows: take a valid 102-avoiding string of length $n-1$ and put a 0 or 2 in the first position to get a length n string that also does not contain 102. There are $2g(n-1)$ such strings. We can also take a valid length $n-1$ string and put a 1 in the first position, but if there is a 02 in the next two positions, we no longer have a valid string. There are $g(n-3)$ valid, length $n-1$ strings starting with 02, so subtracting those out, we have $g(n-1) + g(n-3)$ such strings starting with a 1.

5) $h(n) = 4h(n-1) - 2h(n-2) + h(n-3)$. $h(1) = 4$, $h(2) = 14$, $h(3) = 49$. Our recursion involves looking back 3 terms, ($h(n-3)$) so we need to specify 3 initial values. Valid strings of length n can be formed as follows. Take a valid string of length $n-1$ and place a 0 or 3 in the front. Since the last $n-1$ positions do not contain a 12, placing a 0 or a 3 in the first position preserves this property, and so there are $2h(n-1)$ valid length n strings starting with a 0 or a 3. We can also place a 1 or a 2 in the first position, but then we must subtract out the length $n-1$ strings that have a 2 or a 0 in their first position respectively. The number of valid length $n-1$ strings that have a 2 in the first position is $h(n-2) - h(n-3)$ – it would be $h(n-2)$ but we cannot have a 0 afterward so we subtract out $h(n-3)$ such strings – and hence, the number of valid length n strings starting with a 1 is $h(n-1) - (h(n-2) - h(n-3)$. The number of valid length $n-1$ strings that have a 0 in the first position is $h(n-2)$. Hence, the number of valid length n strings starting with a 2 is $h(n-1) - h(n-2)$. Adding these disjoint cases together gets the result.

7) $gcd(827, 249) = 1$. $a = -168$, $b = 558$. The steps in your algorithm should look as follows:

$$
\begin{aligned}
827 &= 3 \cdot 248 + 80 \\
249 &= 3 \cdot 80 + 9 \\
80 &= 8 \cdot 9 + 8 \\
9 &= 1 \cdot 8 + 1 \\
8 &= 8 \cdot 1 + 0
\end{aligned}
$$

9) **(a)** The base case:

$$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}.$$

For the inductive step, use the inductive hypothesis to conclude

$$1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \frac{(n-1)(n)(2(n-1)+1)}{6} + n^2.$$

Basic algebraic manipulations show that the right hand side is equal to $n(n+1)(2n+1)/6$. For a combinatorial proof, both sides count the number of 3-tuples (x, y, z) where $0 \leq x, y < z \leq n$.

On the left hand side, if $z = k$, then there are k^2 choices for x and y – they can each be any number $0, 1, \ldots, k-1$. Summing up these cases gives the left hand side. For the right hand side, we divide the problem into two cases. **Case 1:** We first consider such triples where $x = y$. There are $\binom{n+1}{2}$ such triples since we choose 2 distinct elements from $\{0, 1, \ldots, n\}$, let the larger number be z and the smaller number be x and y. **Case 2:** If $x < y$, then there are $\binom{n+1}{3}$ such triples. If $x > y$, there are also $\binom{n+1}{3}$ such triples. Basic algebraic manipulation shows that

$$\binom{n+1}{2} + 2\binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6}$$

(b) See Example 2.14 for a combinatorial proof. For the inductive proof, we will use Pascal's formula which we recall for the reader:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Now, for the base case,

$$\binom{1}{0}2^0 + \binom{1}{1}2^1 = 1 + 2 = 3^1.$$

For the inductive step,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}2^k &= \binom{n}{0} + \sum_{k=1}^{n-1}\left(\binom{n-1}{k-1} + \binom{n-1}{k}\right)\cdot 2^k + \binom{n}{n}2^n \\ &= 1 + 2\sum_{k=1}^{n-1}\binom{n-1}{k-1}2^{k-1} + \sum_{k=1}^{n-1}\binom{n-1}{k}2^k + 1 \\ &= 2\cdot 3^{n-1} + 3^{n-1} = 3^n.\end{aligned}$$

11) We show this by induction. For the base case of $n = 0$, $2^0 = 1 = 2^{0+1} - 1$. For the inductive step,

$$\sum_{i=0}^{n} 2^i = 2^n + \sum_{i=0}^{n-1} 2^i = 2^n + 2^n - 1 = 2^{n+1} - 1$$

13) For $n = 1$, $9 - 5 = 4$. For the inductive step,

$$9^n - 5^n = 9\cdot 9^{n-1} + 5\cdot 5^{n-1} = 4\cdot 9^{n-1} + 5(9^{n-1} + 5^{n-1}).$$

By the induction hypothesis, $9^{n-1} + 5^{n-1} = 4k$ for some positive integer k. Hence,

$$9^n - 5^n = 4(9^{n-1} + 5k)$$

which proves the statement.

15) For $n = 1$,

$$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 = 4\cdot 9.$$

For the inductive step, first observe that

$$(n-1)^3 = n^3 - 3n^2 + 3n - 1.$$

Now,

$$\begin{aligned} n^3 + (n+1)^2 + (n+2)^3 &= n^3 + (n+1)^3 + n^3 + 6n^2 + 12n + 8 \\ &= n^3 + (n+1)^3 + n^3 - 3n^2 + 3n - 1 + 9n^2 + 9n + 9 \\ &= (n-1)^3 + n^3 + (n+1)^3 + 9n^2 + 9n + 9 \end{aligned}$$

By the induction hypothesis, the sum of the first 3 terms is divisible by 3. The last 3 terms are obviously divisible by 3, and so this proves the statement.

17) For $n = 0$, $3 \cdot 0^2 - 0 + 2 = 2 = f(0)$, and for $n = 1$, $3 \cdot 1^2 - 1 + 2 = 4 = f(1)$. For the inductive step, since

$$f(n) = 2f(n-1) - f(n-2) + 6,$$

we may apply the inductive hypothesis (we use strong induction here) to conclude

$$f(n) = 2\left(3(n-1)^2 - (n-1) + 2\right) - \left(3(n-2)^2 - (n-2) + 2\right) + 6.$$

Basic algebraic manipulations show that this is equal to $3n^2 - n + 2$.

19) For $n = 0$, $(1+x)^0 = 1 \geq 1 + 0 \cdot x = 1$. For the inductive step,

$$\begin{aligned} (1+x)^n = (1+x)(1+x)^{n-1} &\geq (1+x)(1+(n-1)x) \\ &= 1 + (n-1)x + x + (n-1)x^2 \\ &= 1 + nx + (n-1)x^2 \\ &\geq 1 + nx. \end{aligned}$$