1 Answers to Chapter 8, Odd-numbered Exercises

1) **(a)** $1 + 4x + 6x^2 + 4x^3 + x^4$.

(b) $1 + x + x^2 + x^3 + x^4 + x^7$.

(c) $x^3 + 2x^4 + 3x^5 + 4x^6 + 6x^7$.

(d) $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$.

(e) $3 + x^3 - 4x^4 + 7x^5$.

(f) $x^4 + 2x^5 - 3x^6 + x^8$.

3) **(a)** By the Binomial Theorem, for $0 \le n \le 11$, $a_n = \binom{10}{n}$. For $n > 11$, $a_n = 0$.

(c) For $n = 4k + 3$, $k \ge 0$, $c_n = 1$. Otherwise, $c_n = 0$. The generating function looks like

$$x^3\left(1 + x^4 + x^8 + x^{12} + \ldots\right) = x^3 + x^7 + x^{11} + x^{15} + \ldots + x^{4k+3} + \ldots$$

(e) $e_n = 1$ for $n \ge 0$ except for $n = 3, 4$; $e_3 = e_4 = 2$. We combine the following three generating functions:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$
$$\frac{x^2}{1-x} = \sum_{k=2}^{\infty} x^k$$
$$\frac{x^4}{1-x} = \sum_{k=4}^{\infty} x^k$$

to get that

$$\frac{1 + x^2 - x^4}{1-x} = 1 + x + 2x^2 + 2x^3 + x^4 + x^5 + \ldots + x^k + \ldots.$$

(g) $g_n = (-4)^n$ for $n \ge 0$. To see this,

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n.$$

Now substitute $u = -4x$ into the right hand side and left hand side.

(i)$i_n = 1$ for $n = 7k$ or $n = 7k + 1$ or $n = 7k + 2$ where $k \ge 0$. We combine the following three generating functions:

$$\frac{1}{1-x^7} = \sum_{k=0}^{\infty} x^{7k}$$
$$\frac{x}{1-x^7} = \sum_{k=0}^{\infty} x^{7k+1}$$
$$\frac{x^2}{1-x^7} = \sum_{k=0}^{\infty} x^{7k+2}$$

to get that

$$\frac{x^2+x+1}{1-x^7} = \sum_{k=0} x^{7k} + x^{7k+1} + x^{7k+2}.$$

5) There are 24 ways to get 10 balloons, and the generating function is $\frac{x^2+x^3+x^4}{(1-x)^2}$. Since we want at least one white balloon and at least one gold balloon, we model each of these with the generating function

$$\frac{x}{1-x} = x + x^2 + x^3 + \dots.$$

Since we want at most two blue balloons, we model this with

$$1 + x + x^2.$$

Hence, the number of ways to create a bunch of n balloons is the coefficient of x^n in the expansion of

$$\frac{x}{1-x} \cdot \frac{x}{1-x} \cdot (1+x+x^2) = \frac{x^2+x^3+x^4}{(1-x)^2}.$$

To find this coefficient, observe that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \sum_{n=0}^{\infty}(n+1)x^n.$$

Hence,

$$\begin{aligned}\frac{x^2+x^3+x^4}{(1-x)^2} &= \sum_{n=0}(n+1)x^{n+2} + \sum_{n=0}(n+1)x^{n+3} + \sum_{n=0}(n+1)x^{n+4} \\ &= \sum_{n=2}^{\infty}(n-1)x^n + \sum_{n=3}^{\infty}(n-2)x^n + \sum_{n=4}^{\infty}(n-3)x^n \\ &= x^2 + 3x^3 + \sum_{n=4}^{\infty}(3n-6)x^n\end{aligned}$$

Hence, there are $3n-6$ ways to get a bunch of n balloons.

7) The number of solutions is $\binom{n+1}{3}$. The number of solutions to $x_1+x_2+x_3+x_4 \leq n$ is equal to the number of solutions to $x_1+x_2+x_3+x_4+x_5 = n$. For x_1, x_2, x_3, x_4, x_5, the associated generating functions are, respectively,

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n; \\ \frac{x^2}{1-x} &= \sum_{n=0}^{\infty} x^{n+2} = \sum_{n=2}^{\infty} x^n; \\ \frac{1}{1-x^4} &= \sum_{n=0}^{\infty} x^{4n}; \\ 1+x+x^2+x^3 &= \frac{1-x^4}{1-x}; \\ \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n.\end{aligned}$$

The coefficient of x_n in the product gives us the number of solutions:

$$\frac{1}{1-x} \cdot \frac{x^2}{1-x} \cdot \frac{1}{1-x^4} \cdot \frac{1-x^4}{1-x} \cdot \frac{1}{1-x} = \frac{x^2}{(1-x)^4}$$

Now, since

$$\frac{d^3}{dx^3}\frac{1}{1-x} = \frac{6}{(1-x)^4}$$

we have that

$$\frac{x^2}{(1-x)^4} = \frac{x^2}{6} \cdot \frac{d^3}{dx^3}\frac{1}{1-x} = \frac{x^2}{6}\sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3} = \sum_{n=2}^{\infty}\binom{n+1}{3}x^n.$$

9) $(1+x)^p$. For each of the p students, a student can either be chosen or not. Hence, their associated generating function is $1+x$. Therefore, the generating function for the number of ways to choose n students from p students is

$$(1+x)^p = \sum_{n=0}^{p}\binom{p}{n}x^n.$$

11) 103. For dollar coins, dollar bills, and \$2 bills, the associated generating functions are, respectively:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$$

So, the number of ways to make change for \$100 is the coefficient of x^{100} in the expansion of

$$\frac{1}{(1-x)^2(1-x^2)}.$$

By a partial fraction expansion,

$$\frac{1}{(1-x)^2(1-x^2)} = \frac{1/4}{1-x} + \frac{1/4}{1+x} + \frac{1/2}{(1-x)^2} = \frac{1}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}(n+1)x^n.$$

Hence, the coefficient of x^{100} is $1 + (-1)^{100} + 101 = 103$.

13) **(a)**

$$\sum_{n=0}^{\infty} b_n x^n = \frac{x^2(1-x^7)(1+4x+10x^2)}{(1-x)^2(1-x^2)}.$$

The generating functions for chocolate bites, peanut butter cups, and peppermint candies are, respectively,

$$\frac{x^2}{1-x} = \sum_{n=2}^{\infty} x^n$$
$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$$
$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 = \frac{1-x^7}{1-x}$$

For fruit chews, the generating function is a bit more complicated. If there are no fruit chews in the bag, there is exactly 1 way to do this. If there is 1 fruit chew in the bag, since there are 4 flavors, there are 4 ways to do this. If there are 2 fruit chews in the bag, there are $\binom{4}{2} + 4 = 6 + 4 = 10$ ways to do this (there are $\binom{4}{2}$ ways if the flavors are different, and 4 ways if the flavors are the same). So the generating function for fruit chews is

$$1 + 4x + 10x^2.$$

The product of these is the generating function for b_n.

(b) We are looking for the smallest n such that $b_n \geq 400$. We use a computer algebra system (Wolfram Alpha) to find the partial fraction decomposition:

$$\frac{x^2(1-x^7)(1+4x+10x^2)}{(1-x)^2(1-x^2)} = 266 + 217x + 162x^2 + 118x^3 + 78x^4 + 49x^5 + 24x^6 + 10x^7 - $$
$$- \frac{1281/4}{1-x} + \frac{7/4}{1+x} + \frac{105/2}{(1-x)^2}$$

So for $n > 7$, the coefficient of x^n is

$$\frac{-1281}{4} + (-1)^n \frac{7}{4} + \frac{105}{2}(n+1).$$

Plugging this into a calculator, we see that this is more than 400 for $n \geq 13$. So, 13 pieces of candy are needed.

15) **(a)** 213. The generating functions for pennies, nickels, dimes, and quarters are, respectively,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1-x^5} = \sum n = 0^{\infty} x^{5n}$$
$$\frac{1}{1-x^{10}} = \sum_{n=0}^{\infty} x^{10n}$$
$$\frac{1}{1-x^{25}} = \sum_{n=0}^{\infty} x^{25n}.$$

The coefficient of x^{95} in the expansion of

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

is the number of ways to make change for \$0.95. Using a computer algebra system, we find that this is 213.

(b) The generating functions for pennies, nickels, dimes, and quarters are, respectively,

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$
$$1 + x^5 + \frac{x^{10}}{2!} + \frac{x^{15}}{3!} + \ldots + \frac{x^{5n}}{n!} + \ldots$$
$$1 + x^{10} + \frac{x^{20}}{2!} + \frac{x^{30}}{3!} + \ldots + \frac{x^{10n}}{n!} + \ldots$$
$$1 + x^{25} + \frac{x^{50}}{2!} + \frac{x^{75}}{3!} + \ldots + \frac{x^{25n}}{n!} + \ldots$$

One can then use a computer algebra system, cutting off any terms after x^{95} to get the result.

17) A partition of 10 into odd parts consists of 1s, 3s, 5s, 7s, and 9s. The generating function for each of these is, respectively,

$$1 + x + x^2 + x^3 + \ldots + x^{10}$$
$$1 + x^3 + x^6 + x^9$$
$$1 + x^5 + x^{10}$$
$$1 + x^7$$
$$1 + x^9.$$

Hence, the coefficient of x^{10} in the product of these gives the answer. Multiplying these polynomials together with a computer algebra system yields 10 as the coefficient of x^{10}.

19)

$$\prod_{n=1}^{\infty} \frac{1}{1 - x^{2n}}$$

21) **(a)** $a_n = 7^n$, because

$$e^{7x} = 1 + 7x + \frac{(7x)^2}{2!} + \frac{(7x)^3}{3!} + \ldots$$

(b) $b_n = n(n-1)3^n$, because

$$x^2 e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^{n+2}}{n!} = \sum_{n=2}^{\infty} \frac{(3x)^n}{(n-2)!} = \sum_{n=2}^{\infty} \frac{n(n-1)(3x)^n}{n!}$$

(c) $c_n = n!$.

23) $\frac{1}{2}(4^n - 3^n - 2^n + 1)$. The exponential generating functions for the letters a, b, c, d are, respec-

tively,

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The coefficient of $x^n/n!$ in the product gives the number of strings:

$$(e^x - 1)(e^x)(\frac{e^x - e^{-x}}{2})(e^x) = (e^{3x} - e^{2x})(\frac{e^x - e^{-x}}{2}) = \frac{1}{2}(e^{4x} - e^{3x} - e^{2x} + e^x)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(4^n - 3^n - 2^n + 1)x^n}{n!}.$$

25) $a_0 = 0, a_1 = 0, a_2 = 0$ and for $n \geq 3$,

$$a_n = n2^n - 2n + (-1)^{n-1} \cdot 2n - (n)(n-1) + (-1)^n n(n-1)$$

The exponential generating functions for the letters a, b, c, d are, respectively,

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$
$$\frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$x + \frac{x^2}{2}.$$

The coefficient of $x^n/n!$ in the product gives the number of strings:

$$(e^x - 1)(\frac{e^x - e^{-x}}{2})(e^x)(x + \frac{x^2}{2}) = \frac{1}{4}(e^{2x} - 1 - e^x + e^{-x})(2x + x^2)$$
$$= 2xe^{2x} - 2x - 2xe^x + 2xe^{-x} + x^2e^{2x} - x^2 - x^2e^x + x^2e^{-x}$$
$$= -2x - x^2 + 2\sum_{n=1}^{\infty} \frac{2^{n-1}x^n}{(n-1)!} - 2\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$
$$+ 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(n-1)!} + -\sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} + \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{(n-2)!}$$