Solutions

Student Name and ID Number

MATH 3012 Quiz 1, September 18, 2003, WTT

1. On planet Gorp, DNA is represented by strings of letters from the following six letter alphabet $\{C, D, S, T, X, Z\}$.

a. How many DNA strings of length 17 can be formed?

There are 17 positions, each of which can be occupied by any one of 6 symbols. By the product rule, the answer is:

$$6^{17}$$

b. How many DNA strings of length 17 have exactly 4 C's?

There are $\binom{17}{4}$ ways to select the positions for the C's. There are 13 other positions, each of which can be any of the remaining 5 symbols. By the product rule, the answer is:

$$\binom{17}{4}5^{13}$$

c. How many DNA strings of length 17 contain 4 C's, 2 T's, 8 X's and 3 Z's?

This is just a multinomial coefficient. The answer is:

$$\binom{17}{4, 2, 8, 3}$$

d. Of the strings described in part c, how many have all 4 C's before the two T's?

Consider the six positions occupied by the C's and T's to be a new symbol, say W, and this new symbol will appear six times. We then have a string of 17 symbols with 6 W's, 8 X's and 3 Z's. The number of such strings is again a multinomial coefficient:

$$\binom{17}{6, 8, 3}$$

e. Of the strings described in part d, how many have the 4 C's and the 2 T's occuring together as a block of six consecutive characters?

Now we consider the block as a single character:

$$\binom{12}{1, 8, 3}$$

2. How many integer value solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 + x_4 = 74$, all $x_i > 0$.

Consider 74 objects in a row. There are 73 gaps, and we choose 3 gaps to obtain a partition of 74 into 4 parts, with all parts positive. Therefore, the answer is:

$$\binom{73}{3}$$

b. $x_1 + x_2 + x_3 + x_4 < 74$, all $x_i > 0$.

Add a new variable x_5 with $x_5 > 0$ to obtain the equation $x_1 + x_2 + x + 3 + x_4 + x_5 = 74$. Then apply the same reasoning as in part (a) to obtain:

$$\binom{73}{4}$$

c. $x_1 + x_2 + x_3 + x_4 = 74$, all $x_i \geq 0$.

Add one to each part to make them positive. Sum is now 78. There are 77 gaps and we choose 3. Answer is:

$$\binom{77}{3}$$

d. $x_1 + x_2 + x_3 + x_4 \leq 74$, all $x_i \geq 0$.

Add new variable $x_5 \geq 0$. As before, add one to each variable to make them positive. Sum is now 79, so there are 78 gaps and we choose 4. Answer is then:

$$\binom{78}{4}$$

e. $x_1 + x_2 + x_3 + x_4 = 74$, all $x_i > 0$, $x_4 > 8$.

Replace x_4 by $x_4' = x_4 - 8$. Now $x_1, x_2, x_3, x_4' > 0$ and they sum of 66. There are 65 gaps and we choose 3, so the answer is:

$$\binom{65}{3}$$

3. In three space, consider moves from one point with integer coordinates to another formed by adding one of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. How many paths from $(0,0,0)$ to $(5,3,11)$ can formed with such moves?

A path can be considered as a string of letters chosen from the three letter alphabet $\{L, M, R\}$ (left, middle, right). In this case, there will be a total of $19 = 5+3+11$ letters, with exactly 5 L's, 3 M's and 11 R's. So the answer is the multinomial coefficient:

$$\binom{19}{5,3,11}$$

4. Prove by induction: $2 + 5 + 8 + \cdots + (3n-1) = n(3n+1)/2$.

Proof. Consider first the case when $n = 1$. The left hand side consists of a single term, the integer 2. The right hand side is $1(4)/2$ which equals 2. So the formula holds when $n = 1$. Now assume it is valid when $n = k$, where k is some positive integer, i.e., assume

$$2 + 5 + 8 + \cdots + (3k-1) = \frac{k(3k+1)}{2}.$$

Then it follows that

$$\begin{aligned} 2 + 5 + 8 + \cdots + (3k-1) + (3k+2) &= \frac{k(3k+1)}{2} + (3k+2) \\ &= \frac{3k^2+k}{2} + \frac{6k+4}{2} \\ &= \frac{3k^2+7k+4}{2} \\ &= \frac{(k+1)(3k+4)}{2} \end{aligned}$$

This last equation shows that the formula also holds when $n = k+1$. Therefore, by the Principle of Induction, it holds for all positive integers.

5. Use the Euclidean algorithm to find $d = \gcd(90, 336)$.

By long division, $336 = 3 \times 90 + 66$; $90 = 1 \times 66 + 24$; $66 = 2 \times 24 + 18$; $24 = 1 \times 18 + 6$; and $18 = 3 \times 6 + 0$. Therefore, $\gcd(90, 336) = 6$.

Then find integers x and y so that $d = 90x + 336y$.

Solving for the remainders, we have $6 = 24 - 18$; $18 = 66 - 2 \times 24$; $24 = 90 - 66$; and $66 = 336 - 3 \times 90$. So

$$\begin{aligned} 6 &= 24 - 18 = 24 - (66 - 2 \times 24) = -66 + 3 \times 24 \\ &= -66 + 3(90 - 66) = 3 \times 98 - 4 \times 66 \\ &= 3 \times 90 - 4 \times (336 - 3 \times 90) = -4 \times 336 + 15 \times 90 \end{aligned}$$

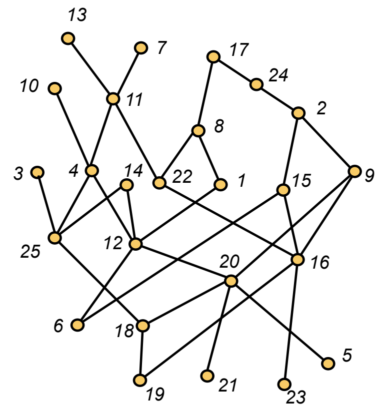
So $x = 90$ and $y = -4$ will work.

6. Find the coefficient of $x^8 y^4 z^{11}$ in $(2x + 3y - 5z)^{23}$.

By the binomial theorem, the answer is:

$$\binom{23}{8, 4, 11} 2^8 3^4 (-5)^{11}$$

7. Consider the partially ordered set (poset) shown below:



a. Find the set of maximal elements.

$$A = \{3, 10, 7, 13, 14, 17\}$$

b. Find a minimum partition of this poset into antichains.

We recursively remove the maximal elements to obtain:

$$\begin{aligned} A_1 &= \{3, 10, 7, 13, 14, 17\} \\ A_2 &= \{8, 11, 24\} \\ A_3 &= \{1, 2, 4, 22\} \\ A_4 &= \{9, 12, 15, 25\} \\ A_5 &= \{6, 16, 20\} \\ A_6 &= \{5, 18, 21, 23\} \\ A_7 &= \{19\} \end{aligned}$$

c. Find the height h of this poset (maximum number of points in a chain).

The height h is the number of antichains in the partition above; so $h = 7$.

d. Find a chain of h points in this poset.

Starting with 19, we backtrack up through the antichains and find the chain $\{19, 18, 20, 9, 2, 24, 17\}$.

e. Find a maximal antichain containing 4 elements.

This is the *most difficult* problem on the test—from both from the student and professor perspectives, i.e., it is difficult to answer and it is difficult to grade. Looking over the diagram, I found the following answers:

$$\{6, 16, 20, 25\}, \quad \{2, 12, 22, 25\}, \quad \{3, 4, 14, 17\}$$

But there may be more! Now I could write a computer program to find them all. How hard would this be?

f. Explain why this poset cannot be partitioned into 6 chains.

It is easy to see that

$$\{3, 4, 14, 22, 1, 15, 9\}$$

is an antichain of 7 points. If there were a partition of the poset into 6 chains, then by the Pigeon Hole Principle, some two of these seven points would have to belong to the same chain, which is impossible.