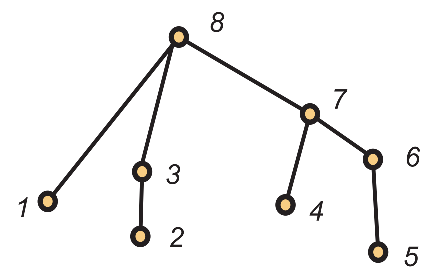
Student Name and ID Number

MATH 3012 Quiz 3, November 25, 2003, WTT

1. Draw a tree with vertex set $\{1, 2, \ldots, 8\}$ so that if the vertices are colored in natural order using First Fit, then 4 colors will be used.

There are many such trees. Here is one of them.



2. Consider the graph G with non-negative edge weights given in the following text file shown below on the left (note that the file has been sorted by edge weights). In the space to the right List the edges *in order* which result from applying Kruskal's algorithm (avoid cycles) to find a minimum weight spanning tree of G. Then set vertex 1 to be the root and list *in order* the edges which result from applying Prim's algorithm (build tree).

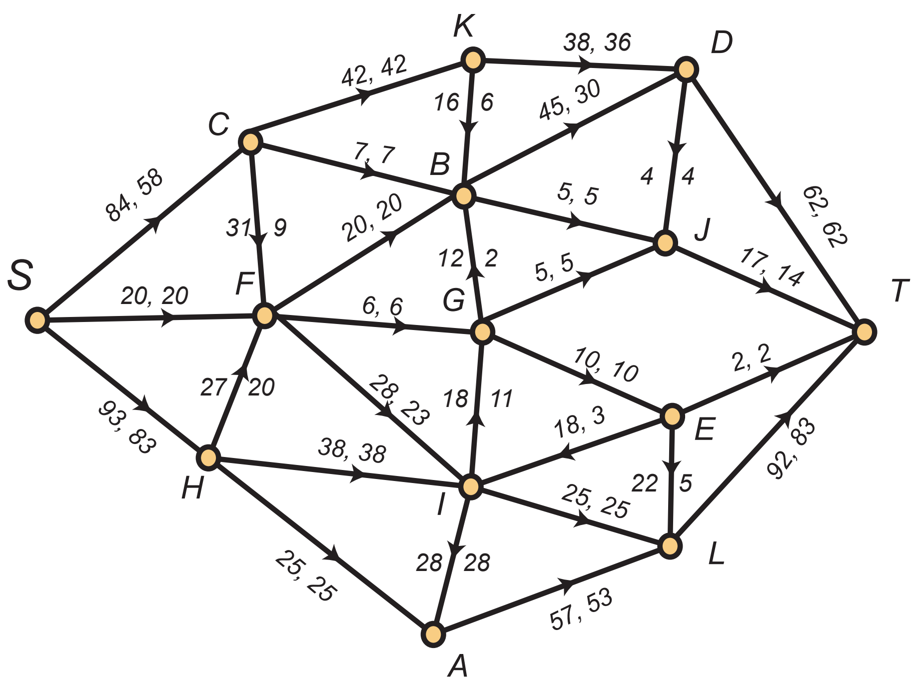
`graph1.txt`	**Kruskal**	**Prim**
3 6 19	3 6 19	1 6 36
2 4 22	2 4 22	3 6 19
5 6 24	5 6 24	5 6 24
3 5 26	2 3 28	2 3 28
2 6 27	1 6 36	2 4 22
2 3 28		
2 5 29		
1 6 36		
1 2 41		
1 5 45		
2 6 51		

3. The matrix given (below left) is the distance matrix for a digraph D whose vertex set is $\{1, 2, 3, 4, 5, 6\}$. In the space to the right, apply Dijkstra's algorithm to find all the shortest paths from vertex 1 to all other vertices in the digraph D.

D	1	2	3	4	5	6
1	0	10	50	20	32	4
2	82	0	29	7	18	5
3	47	62	0	18	8	62
4	15	44	16	0	8	13
5	22	10	6	12	0	27
6	19	5	81	14	29	0

Scan	vertex	2	3	4	5	6
1	distance	10	50	20	32	4
	path	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
6	distance	9	50	18	32	4
	path	(1,6,2)	(1,3)	(1,6,4)	(1,5)	
2	distance		38	16	27	
	path		(1,6,2,3)	(1,6,2,4)	(1,6,2,5)	
4	distance		32		24	
	path		(1,6,2,4,3)		(1,6,2,4,5)	
5	distance		30			
	path		(1,6,2,4,5,3)			

4. Consider the following network flow problem.



a. What is the current value of the flow?

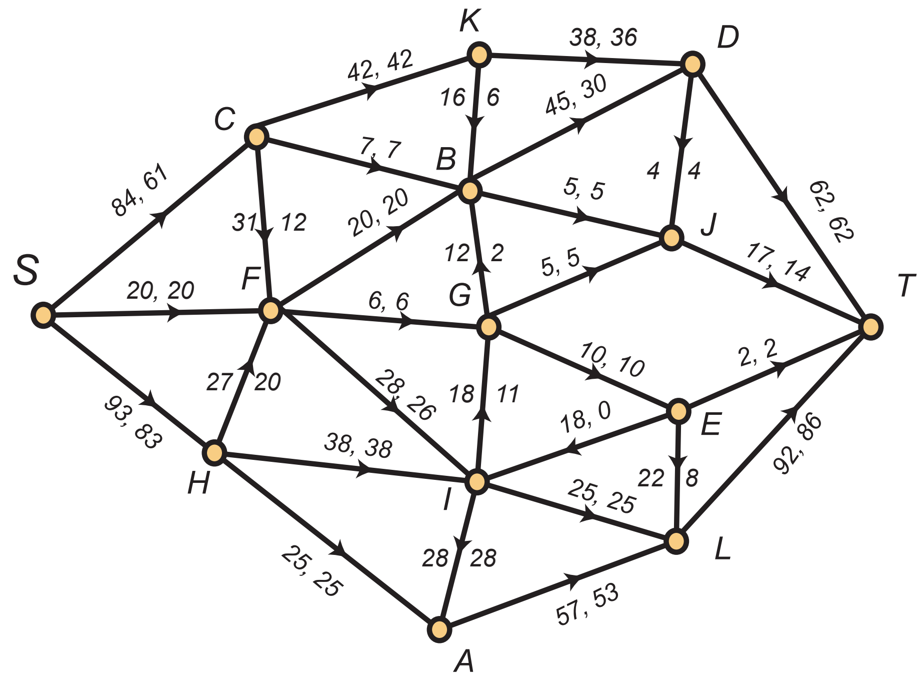
$$\text{value} = 58 + 20 + 83 = 161.$$

b. What is the capacity of the cut: $L = \{S, A, B, C, E, F, G, H, I, L\}$; $U = \{D, J, K, T\}$?

$$\text{capacity} = 42 + 45 + 5 + 5 + 2 + 92 = 191$$

c. List *in order* the labels given to vertices labelling algorithm using alphabetic ordering on the nodes and find an augmenting path. Then increase the flow by the binding value of this path, recording the changes directly on the diagram.

$S \quad (*, +, \infty)$
$C \quad (S, +, 26)$
$H \quad (S, +, 10)$
$F \quad (C, +, 22)$
$I \quad (F, +, 5)$
$E \quad (I, -, 3)$
$G \quad (I, +, 5)$
$L \quad (E, +, 3)$
$B \quad (G, +, 5)$
$A \quad (L, -, 3)$
$T \quad (L, +, 3)$



d. Repeat the labelling algorithm on the new network flow. It will halt with the sink unlabelled. Find a saturated cut.

$S \quad (*, +, \infty)$
$C \quad (S, +, 23)$
$H \quad (S, +, 10)$
$F \quad (C, +, 19)$
$I \quad (F, +, 2)$
$G \quad (I, +, 2)$
$B \quad (G, +, 2)$
$D \quad (B, +, 2)$
$K \quad (B, +, 2)$

So the labelled vertices are $L = \{S, C, H, F, I, G, B, D, K\}$ and the unlabelled vertices are $U = \{T, A, E, J, L\}$. These two sets form a cut with capacity $62 + 4 + 5 + 5 + 10 + 25 + 28 + 25 = 164$, the value of the current flow. This shows that the flow is maximum.

5. (Extra Credit) Prove the correctness of Dijkstra's algorithm

Proof. Let D be a digraph with positive weights on the edges. Also, let r denote the root. We show that for every vertex x in D, Dijkstra's algorithm finds a minimum length path from r to x. The argument is by induction on the minimum number t of edges in a shortest length path from r to x. If $t = 0$, then $r = x$ and Dijkstra's algorithm correctly reports the optimal path as having length 0. If $t = 1$, then when we scan from the root r, we will consider the edge (r, x) which would then be an optimal path. Thus Dijkstra's algorithm will report the length of this edge as the optimal path from r to x.

Now suppose the result holds when $t \leq k$ and consider the case $t = k + 1$. Now let $P = (u_0, u_1, u_2, \dots, u_t, u_{t+1})$ be a shortest path from $r = u_0$ to $u_{t+1} = x$. Then the initial segment $P' = (u_0, u_1, u_2, \dots, u_t)$ is a shortest path from r to u_t. By induction, Dijkstra finds a path of this length from r to u_t and then scans from u_t. During this scan, it sees the edge (u_t, u_{t+1}) and thus reports a path from r to x which has at most the length of P' plus the weight on the edge (u_t, u_{t+1}). But this is the length of P.

6. (Extra Credit) State and prove the key lemma used to show that Kruskal's and Prim's algorithms find minimum spanning trees.

Lemma. Let $G = (V, E)$ be a connected graph with non-negative weights on the edges, and let $\mathcal{F}$ be a spanning forest. Let C be an arbitrary component of $\mathcal{F}$ and among all edges of G with exactly one end point in C, let e be an edge of mininum weight. Then among all spanning trees of G which contain $\mathcal{F}$, there is one of minimum weight which also contains the edge e.

Proof. We argue by contradiction. Suppose the lemma is false and let T be a minimum weight spanning tree among those which contain the edges of $\mathcal{F}$. Then the edge e does not belong to T. Label the two vertices at the ends of e as x and y with x in component C. Since T is a spanning tree, there is a unique path P from x to y in T. The edges in P plus the edge e form a cycle in G. Proceeding along the path P, let f be the first edge with one end point in C and the other not in C. Then let T' be the tree formed by removing f from T and replacing it with e.

By the definition of e, we know that f has weight at least as large as the weight of e. Thus the weight of T' is at most as large as the weight of T. But T' contains e as well as all the edges of $\mathcal{F}$. The contradiction completes the proof.