Super Student

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MATH 3012 Quiz 1, September 16, 2004, WTT

1. Consider the family of all strings of length 20 formed from the seven letter alphabet $\{A, B, C, D, E, F, G\}$.

a. What is the total number of strings?

A string of 20 letters; 7 choices for each. Total is 7^{20}.

b. How many strings have exactly 4 A's?

First choose the positions for the four A's. In the other sixteen positions, we have six choices. Answer is $\binom{20}{4}6^{16}$.

c. How many strings have exactly four A's, five B's, eight C's and three D's?

This is the multinomial coefficient $\binom{20}{4,5,8,3}$. Another way to express this is $\frac{20!}{4!5!8!3!}$.

d. Of the strings described in part c, how many have all five B's before the three D's?

Consider the eight positions occupied by B's and C's as a single *new* letter, say X. So the answer is $\binom{20}{4,8,8}$, or $\frac{20!}{4!8!8!}$.

e. Of the strings described in part d, how many have the five B's and the three D's occuring together as a block of eight consecutive characters?

Now these eight positions become a single letter. So the answer is $\binom{13}{1,4,8}$, or $\frac{13!}{1!4!8!}$.

2. How many integer value solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 = 54$, all $x_i > 0$.

53 gaps; choose two. Answer is $\binom{53}{2}$.

b. $x_1 + x_2 + x_3 = 54$, all $x_i \geq 0$.

Add three artificial apples. 56 gaps; choose two. Answer is $\binom{56}{2}$.

c. $x_1 + x_2 + x_3 < 54$, all $x_i \geq 0$.

Same as $x_1 + x_2 + x + 3 <= 53$. Now add x_4 to make an equation: $x_1 + x_2 + x_3 + x_4 = 53$. Apply same reasoning as in part b. Answer is $\binom{56}{3}$.

d. $x_1 + x_2 + x_3 \leq 54$, all $x_i \geq 0$.

Add x_4 to make an equation: $x_1 + x_2 + x + 3 + x_4 = 54$. Apply same reasoning as in part b. Answer is $\binom{57}{3}$.

e. $x_1 + x_2 + x_3 = 74$, all $x_i > 0$, $x_3 > 7$.

Take off 7 in advance for x_3. Then we have $x_1 + x_2 + x_3' = 67$ with x_1, x_2 and x_3' all positive. Answer is $\binom{66}{2}$.

3. In three space, consider moves from one point with integer coordinates to another formed by adding one of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. How many paths from $(0, 0, 0)$ to $(4, 7, 13)$ can formed with such moves?

A total of 24 moves, 4 of which use coordinate 1, 7 use coordinate 2 and 13 use coordinate 3. Answer is $\binom{24}{4,7,13}$, or $\frac{24!}{4!}7!13!$.

4. Prove by induction: $1 + 4 + 7 + \cdots + (3n - 2) = n(3n - 1)/2$.

When $n=1$, the left hand side is $3\cdot1-2=1$, while the right hand side is $1(3\cdot1-1)/2=1$. Therefore the formula is valid when $n=1$. Now assume the formula is valid when $n=k$, where k is some positive integer, i.e.,

$$1+4+7+\cdots+(3k-2)=k(3k-1)/2.$$

Then it follows that:

$$\begin{aligned}
1+4+7+\cdots+(3k-2)+[3(k+1)-2] &= k(3k-1)/2+[3(k+1)-2]\\
&= \frac{3k^2-k}{2}+3k+1\\
&= \frac{3k^2-k}{2}+\frac{6k+2}{2}\\
&= \frac{3k^2+5k+2}{2}\\
&= \frac{(k+1)(3k+2)}{2}\\
&= \frac{(k+1)[3(k+1)-1)}{2}
\end{aligned}$$

This shows that the formula also holds when $n=k+1$. By the principle of induction, it then holds for every positive integer n.

5. Use the Euclidean algorithm to find $d=\gcd(2340,924)$.
By repeated long division:

$$\begin{aligned}
2340 &= 2\times 924+492\\
924 &= 1\times 492+432\\
492 &= 1\times 432+60\\
432 &= 7\times 60+12\\
60 &= 5\times 12+0
\end{aligned}$$

It follows that $\gcd(2340,924)=12$.

Then find integers x and y so that $d=2340x+924y$.
Rewrite the equations as:

$$\begin{aligned}
492 &= 2340-2\times 924\\
432 &= 924-1\times 492\\
60 &= 492-1\times 432\\
12 &= 432-7\times 60
\end{aligned}$$

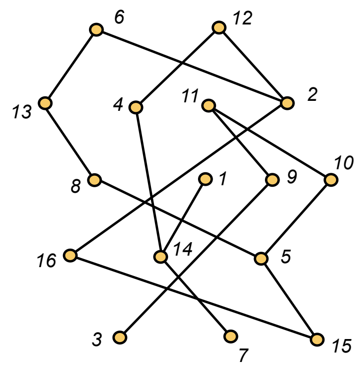
Substitute to obtain

$$\begin{aligned}
12 &= 432-7\times 60\\
&= 1\times 432-7\times(1\cdot 492-1\cdot 432)\\
&= 8\cdot 432-7\cdot 492\\
&= 8(1\cdot 924-1\cdot 492)-7\cdot 492\\
&= 8\cdot 924-15\cdot 492\\
&= 8\cdot 924-15(2340-2\cdot 924)\\
&= 38\cdot 924-15\cdot 2340
\end{aligned}$$

6. Find the coefficient of $x^4y^8z^{23}$ in $(3x-2y-7z)^{35}$.

$$\binom{35}{4,8,23}3^4(-2)^8(-7)^{23}$$

7. Consider the partially ordered set (poset) shown below:



a. Find the set of maximal elements.

$$\text{MAX}(P) = \{6, 12, 11, 1\}$$

b. Find the height h of this poset and find a partition into h antichains.

$$A_1 = \{6, 12, 11, 1\}$$
$$A_2 = \{13, 4, 2, 9, 10\}$$
$$A_3 = \{8, 14, 16, 3\}$$
$$A_4 = \{5, 7\}$$
$$A_5 = \{15\}$$

So the height h of this poset is 5.

c. Find a chain of h points in this poset.

Starting with the point 15 from A_5 and back-tracking, we find the maximum chain:

$$C = \{15, 5, 8, 13, 6\}$$

d. Find an antichain containing 6 points.

Here is one (there may be several more):

$$A = \{4, 16, 8, 1, 9, 10\}$$

e. Show that the width of the poset is 6 by finding a partition into 6 chains.

Here is a partition into six chains. There are many other ways to accomplish this same goal.

$$C_1 = \{6, 13, 8, 5, 15\}$$
$$C_2 = \{12, 4, 14, 7\}$$
$$C_3 = \{2, 16\}$$
$$C_4 = \{11, 10\}$$
$$C_5 = \{9, 3\}$$
$$C_6 = \{1\}$$

Scoring:

1. Problem 1: Five parts, 2 points each, total 10 points.
2. Problem 2: Five parts, 2 points each, total 10 points.
3. Problem 3: 5 points.
4. Problem 4: 15 points.
5. Problem 5: First part (gcd) 10 points, second part 5 points, total 15 points.
6. Problem 6: 10 points.
7. Problem 7: Five parts, 5 points each, total 25 points.