Student Name and ID Number

MATH 3012 Quiz 2, October 12, 2004, WTT

1. Note that $67375 = 5^3 \times 7^2 \times 11$. Compute $\phi(67375)$.

$$\begin{aligned}\phi(67375) &= 67375(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11}) \\ &= 5^3 \cdot 7^2 \cdot 11 \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \\ &= 5^2 \cdot 7 \cdot 4 \cdot 6 \cdot 10 \\ &= 42000\end{aligned}$$

2. (a) Write all the partitions of the integer 8;

$$\begin{aligned}8 &= 8 \quad \text{distinct parts} \\ &= 7+1 \quad \text{distinct parts, odd parts} \\ &= 6+2 \quad \text{distinct parts} \\ &= 6+1+1 \\ &= 5+3 \quad \text{distinct parts, odd parts} \\ &= 5+2+1 \quad \text{distinct parts} \\ &= 5+1+1+1 \quad \text{odd parts} \\ &= 4+4 \\ &= 4+3+1 \quad \text{distinct parts} \\ &= 4+2+2 \\ &= 4+2+1+1 \\ &= 4+1+1+1+1 \\ &= 3+3+2 \\ &= 3+3+1+1 \quad \text{odd parts} \\ &= 3+2+2+1 \\ &= 3+2+1+1+1 \\ &= 3+1+1+1+1+1 \quad \text{odd parts} \\ &= 2+2+2+2 \\ &= 2+2+2+1+1 \\ &= 2+2+1+1+1+1 \\ &= 2+1+1+1+1+1+1 \\ &= 1+1+1+1+1+1+1+1 \quad \text{odd parts}\end{aligned}$$

(b) Of the partitions listed in part (a), how many use distinct parts?

There are 6 partitions of the integer 8 into distinct parts.

(c) Of the partitions listed in part (a), how many use odd parts?

There are 6 partitions of the integer 8 into odd parts. More generally, for every integer n, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

3. Write the inclusion/exclusion formula for the number of onto functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$.

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$

4. Write the inclusion/exclusion formula for the number of derangements of $\{1, 2, \ldots, n\}$.

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

5. Let A denote the advancement operator, i.e., $Af(n) = f(n+1)$. Find the general solution of the following equation:

$$(2A^2 + 7A - 15)f(n) = 0$$

Note that we can factor the quadratic $2A^2 + 7A - 15$ as $(2A-3)(A+5)$ so the roots are $\frac{3}{2}$ and -5. Therefore the general solution is $c_1(\frac{3}{2})^n + c_2(-5)^n$.

6. For the equation in the preceding problem, find the particular solution given $f(0) = 6$ and $f(1) = -4$.

Substituting $n = 0$ and $n = 1$ in the formula for the general solution, we obtain the following two equations for c_1 and c_2:

$$\begin{aligned} c_1 + c_2 &= 6 \\ \frac{3}{2}c_1 - 5c_2 &= -4 \end{aligned}$$

The solution to this system is $c_1 = 4$ and $c_2 = 2$. So the answer is then $4(\frac{3}{2})^n + 2(-5)^n$.

7. Find the general solution of the following equation:

$$(A-1)^2(A-3)^4(A-4+i)^3 f(n) = 0$$

$$\begin{aligned} f(n) = {} & c_1 + c_2 n \\ & + c_3 3^n + c_4 n 3^n + c_5 n^2 3^n + c_6 n^3 3^n \\ & + c_7 (4-i)^n + c_8 n (4-i)^n + c_9 n^2 (4-i)^n \end{aligned}$$

8. Let r_n denote the number of regions in the plane determined by n circles—provided each pair of circles intersects in exactly two points. (a) Write a recurrence equation for r_n.

Label the n circles as $C_1, C_2, \ldots, C_n$. Circle C_n intersects each other circle in exactly two points, so there are $2(n-1)$ points of intersection on C_n. These points divide circle C_n into $2(n-1)$ arcs, and each of these arcs divides an "old" region into two "new" ones. So the recursion is

$$r_n = r_{n-1} + 2(n-1)$$

(b) Solve the recurrence equation in part (a).

The general solution to the homogeneous equation $r_n = r_{n-1}$ is $f(n) = c$. We look for a particular solution to the non-homogeneous equation of the form $f(n) = An + Bn^2$. Substituting, we obtain:

$$\begin{aligned} An + Bn^2 =& A(n-1) + B(n-1)^2 + 2(n-1) \\ =& An - A + Bn^2 - 2Bn + B + 2n - 2 \end{aligned}$$

Equating coefficients, we obtain the two equations:

$$\begin{aligned} 2 - 2B =& 0 \\ -A + B =& 2 \end{aligned}$$

Thus $B = 1$ and $A = -1$. So the solution is $f(n) = n^2 - n + c$. Substituting $n = 1$ and noting that $r_1 = 2$, we obtain $2 = f(1) = 1^2 - 1 + c = c$. It follows that the final answer is $f(n) = n^2 - n + 2$.

9. (Extra Credit) Explain how the principle of inclusion/exclusion is used to derive the formula in Problem 3 for the number of onto functions.

Consider the set X of all functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. For each $j = 1, 2, \ldots, n$, we say that a function $f \in X$ satisfies property P_j if j is NOT in the range of f. Now let S be a set of i properties. Then the number of functions from X which satisfy the properties in S is $(n-i)^m$. By the principle of inclusion/exclusion, the number of onto functions is then:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$