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KEY

Student Name and ID Number

MATH 3012 Quiz 1, February 10, 2009, WTT

1. Consider the 16-element set consisting of the ten digits $\{0, 1, 2, \ldots, 9\}$ and the six capital letters $\{A, B, C, D, E, F\}$.

a. How many strings of length 11 can be formed if repetition of symbols is permitted?

16^{11}

(8) $= 4 \times 2$

b. How many strings of length 11 can be formed if repetition of symbols is *not* permitted?

$P(16, 11)$

c. How many strings of length 11 can be formed using exactly three 5's, six A's and two D's?

$\binom{11}{3,6,2} = \frac{11!}{3!\,6!\,2!}$

d. How many strings of length 11 can be formed if exactly three characters are digits and exactly five of the remaining characters are B's?

$\binom{11}{3}\binom{8}{5} 10^3 5^3$

(8)

2. How many lattice paths from $(3, 2)$ to $(23, 17)$ pass through $(9, 6)$?

$\begin{array}{r} 9\ 6 \\ 3\ 2 \\ \hline 6\ 4 \end{array}$ $\quad \begin{array}{r} 23\ 17 \\ 9\ \ 6 \\ \hline 14\ 11 \end{array}$

$\binom{10}{4}\binom{25}{11}$ or $\binom{10}{6}\binom{25}{14}$

3. How many integer valued solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 + x_4 = 59$, all $x_i \geq 0$.

$\binom{62}{3}$

b. $x_1 + x_2 + x_3 + x_4 = 59$, all $x_i > 0$.

$\binom{58}{3}$

(12) $= 6 \times 2$

c. $x_1 + x_2 + x_3 + x_4 < 59$, all $x_i \geq 0$.

$\binom{62}{4}$

d. $x_1 + x_2 + x_3 + x_4 \leq 59$, all $x_i > 0$.

$\binom{59}{4}$

e. $x_1 + x_2 + x_3 + x_4 \leq 59$, all $x_i > 0$, $x_2 \geq 7$.

$\binom{53}{4}$

f. $x_1 + x_2 + x_3 + x_4 + x_5 \leq 59$, all $x_i > 0$, $x_2 \leq 6$.

Typo

Note: I meant the correct answer to be $\binom{59}{4} - \binom{53}{4}$.

But actually it's $\binom{59}{5} - \binom{53}{5}$.

Either answer accepted

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4. Use the Euclidean algorithm to find $d = \gcd(17160, 168)$.

$$\begin{array}{r} 102 \\ 168 \overline{)17160} \\ \underline{168} \\ 360 \\ \underline{336} \\ 24 \end{array} \qquad \begin{array}{r} 7 \\ 24 \overline{)168} \\ \underline{168} \\ 0 \end{array}$$

$\boxed{\gcd(17160, 168) = 24}$

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5. Use your work in the preceding problem to find integers a and b so that $d = 17160a + 168b$.

$17160 = 102 \cdot 168 + 24$

$168 = 7 \cdot 24$

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$24 = 1 \cdot 17160 - 102 \cdot 168$

so $\boxed{a = 1 \quad b = -102}$

6. For a positive integer n, let t_n count the number of ways to tile a $2 \times n$ checkerboard with figures of ~~two~~ five types:

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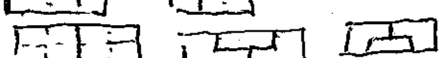
1. A horizontal strip of height 1 and width 2, i.e. a block of size 1×2, one row and two columnns. Such strips can only be oriented horizontally, and not vertically.
2. An "L" shaped region consisting of three 1×1 squares. This figure can be oriented in any of the four possible ways (see drawing on the board).

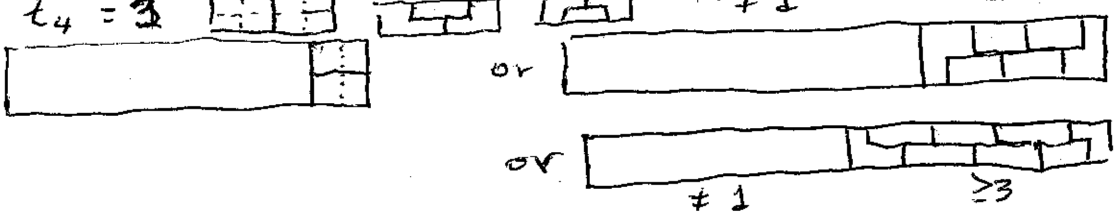
Find a recurrence equation satisfied by t_n and use it to calculate t_8.

$t_1 = 0$

$t_2 = 1$ 

$t_3 = 2$ 

$t_4 = 3$ 



when $n \geq 5$ $\quad t_n = t_{n-2} + 2t_{n-3} + 2t_{n-4} + \dots + 2t_3 + 2t_2 + 2$

$t_5 = t_3 + 2t_2 + 2 = 2 + 2 \cdot 1 + 2 = 6$

$t_6 = t_4 + 2t_3 + 2t_2 + 2 = 3 + 2 \cdot 2 + 2 \cdot 1 + 2 = 11$

$t_7 = t_5 + 2t_4 + 2t_3 + 2t_2 + 2 = 6 + 2 \cdot 3 + 2 \cdot 2 \cdot 2 \cdot 1 + 2$
$= 20$

$t_8 = t_6 + 2t_5 + 2t_4 + 2t_3 + 2t_2 + 2$
$= 11 + 2 \cdot 6 + 2 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 + 2$

$\boxed{t_8 = 37}$

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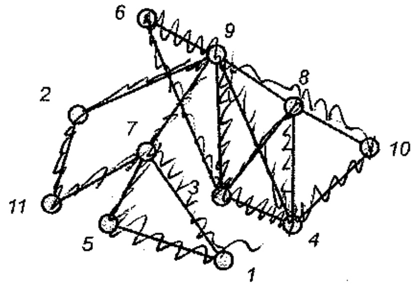
7. Use the algorithm developed in class to find an Euler circuit in the following graph:



(1, 5, 7, 1) (7, 9, 2, 11, 7)

(1, 5, 7, 9, 2, 11, 7, 1) (9, 3, 4, 8, 3, 6, 9, 4, 10, 8, 9)

(1, 5, 7, 9, 3, 4, 8, 3, 6, 9, 4, 10, 8, 9, 2, 11, 7, 1)

8. Consider the following graph:

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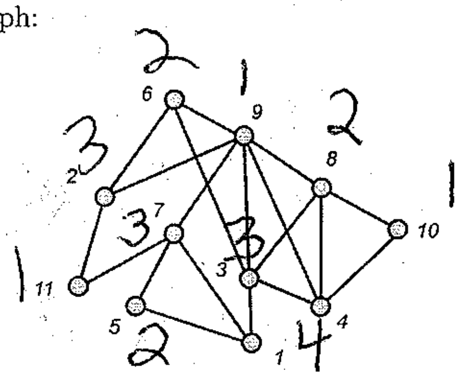
= 6 + 2



a. Explain why this graph does not have an Euler circuit. It has vertices of odd degree such as 6 and 1

b. Provide a listing of the vertices that constitutes a Hamiltonian cycle. (1, 5, 7, 11, 2, 6, 9, 8, 10, 4, 3)

c. Find a set of vertices that forms a maximal clique but not a maximum clique. Many correct answers, e.g. {1, 5, 7}

d. What is $\omega(G)$ for this graph? 4

e. Find a set of vertices which forms a maximum clique in this graph. {3, 4, 8, 9}

f. Show that $\chi(G) = \omega(G)$ for this graph by providing an optimum coloring. You may write directly on the figure.

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9. Prove the following identity by Mathematical Induction:

$$7 + 11 + 15 + \ldots 4n + 3 = 2n^2 + 5n$$

Note: We intend that the expression on the left is just the integer 7 when $n = 1$. Furthemore, when $n \geq 2$, we intend that we are summing up the first n terms in the sequence which begins with $s_1 = 7$ and satisfies $s_n = s_{n-1} + 4$.

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Proof. When $n = 1$, LHS $= 7$ while RHS $= 2 \cdot 1^2 + 5 \cdot 1 = 7$ so the formula is valid when $n = 1$.

Now assume the formula holds when $n = k$ where $k \geq 1$, i.e., we assume

$$7 + 11 + 15 + \cdots + 4k + 3 = 2k^2 + 5k$$

Then

$$\begin{aligned} 7 + 11 + 15 + \cdots + 4k + 3 + 4(k+1) + 3 &= 2k^2 + 5k + [4(k+1) + 3] \\ &= 2k^2 + 5k + 4k + 7 \\ &= 2k^2 + 9k + 7 \\ &= 2k^2 + 4k + 2 + 5k + 5 \\ &= 2(k+1)^2 + 5(k+1) \end{aligned}$$

This shows that the formula also holds when $n = k+1$. Therefore, by the principle of mathematical induction, it holds for all $n \geq 1$

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28 + 33 + 24 + 15 = 100