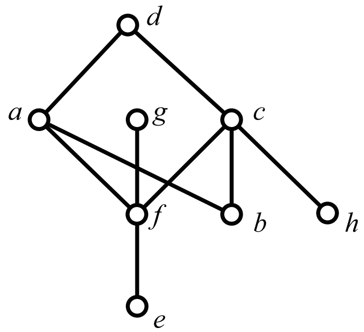
Student Name and ID Number

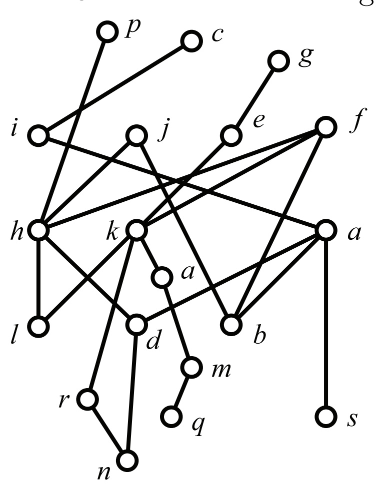
MATH 3012 Quiz 2, March 15, 2013, WTT

1. Consider the poset shown below. The ground set is $X = \{a, b, c, d, e, f, g, h\}$. In the space to the right of the figure, write the reflexive, antisymmetric and transitive relation on X which defines this poset.



$P = \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (g,g), (h,h),$
$(a,d), (c,d), (f,a), (f,g), (f,c), (f,d), (b,a), (b,c), (b,d),$
$(e,f), (e,a), (e,g), (e,c), (e,d)\}$.

2. Consider the following poset.



a. Find all points comparable to k: $\{e, g, f, a, m, q, r, n\}$.

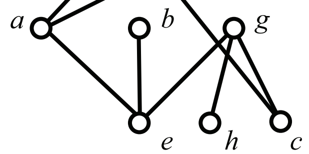
b. Find all points which cover k: $\{e, f\}$.

c. Find a maximal chain of size 2: $\{f, h\}$ or $\{j, h\}$.

d. Using the algorithm taught in class (recursively removing the set of minimal elements), find the height h of the poset and a partition of P into h antichains. Also find a maximum chain. You may indicate the partition by writing directly on the diagam.

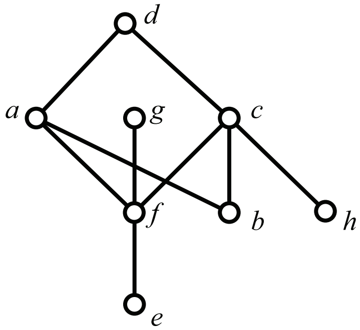
The height h is 6 and $\{g, e, k, a, m, q\}$ is a maximum chain.

3. Find by inspection the width w of the following poset and find a partition of the poset into w chains. Also find a maximum antichain. You may indicate the partition by writing directly on the diagram.



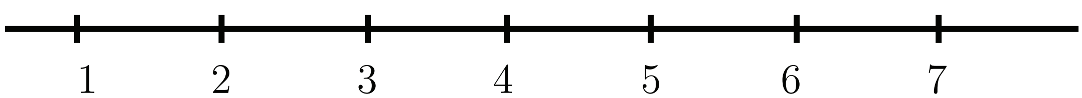
a. The width w is 4 and $\{a, b, c, h\}$ is a maximum antichain.
b. This poset is not an interval order. Find four points which form a copy of $\mathbf{2} + \mathbf{2}$: $\{a, f, g, c\}$, also $\{a, f, g, h\}$.

4. Shown below is the diagram of an interval order. Use the algorithm taught in class to find an interval representation by computing the down-sets and up-sets in the space provided. Then use the First Fit coloring algorithm to find the width w and a partition of the poset into w chains. Also, find a maximum antichain.



$D(a) = b, e, h$
$D(b) = \emptyset$
$D(c) = b, e, f, h$
$D(d) = a, b, c, e, f, h$
$D(e) = \emptyset$
$D(f) = e$
$D(g) = e, f$
$D(h) = \emptyset$

$U(a) = d$
$U(b) = a, c, d$
$U(c) = d$
$U(d) = \emptyset$
$U(e) = a, c, d, f, g$
$U(f) = a, c, d, g$
$U(g) = \emptyset$
$U(h) = c, d$



The width w is 3 and $\{a, c, g\}$ is a maximum antichain (there are several others).

5. Let $\mathbf{2}^{15}$ be the poset consisting of all subsets of $\{1, 2, 3, \ldots, 15\}$, ordered by inclusion.

a. What is the height of this poset: 16.

b. What is the width of this poset: $\binom{15}{7}$, which of course is the same as $\binom{15}{8}$.

c. How many maximal chains does the poset have: 15!.

d. How many maximal chains in this poset pass through the set $\{2, 3, 8, 13\}$: $4! \cdot 11!$.

6. Write the general solution to the homogeneous advancement operator equation:
$\big[A-(7-2i)\big]^3(A-1)^4 f = 0$.
$f(n) = c_1(7-2i)^n + c_2 n(7-2i)^n + c_3 n^2(7-2i)^n + c_4 + c_5 n + c_6 n^2 + c_7 n^3$.

7. Find a particular solution to the advancement operator equation:
$(A^2-3A+5)f = 4\cdot 3^n$.
We try $f(n) = c3^n$. This requires:

$$\begin{aligned} 4\cdot 3^n &= (A^2-3A+5)c3^n \\ &= c3^{n+2} - 3c3^{n+1} + 5c3^n \\ &= 9c3^n - 9c3^n + 5c3^n \\ &= 5c3^n \end{aligned}$$

This implies $4 = 5c$ so that $c = 4/5$ and $f(n) = 4/53^n$ is a solution. **8.** Write the inclusion-exclusion formula for $S(n,m)$, the number of surjections from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,m\}$. Then use this formula to calculate $S(6,4)$.
$S(n,m) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n$.

$$\begin{aligned} S(6,4) &= \sum_{k=0}^{4}(-1)^k\binom{4}{k}(4-k)^6 \\ &= \binom{4}{0}4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6 + \binom{4}{4}0^6 \\ &= 1\cdot 4096 - 4\cdot 729 + 6\cdot 64 - 4\cdot 1 \\ &= 1560. \end{aligned}$$

9. Write the inclusion formula for the number d_n of derangements of $\{1,2,\ldots,n\}$. Then use this formula to calculate d_6.
$d_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)!$

$$\begin{aligned} d_6 &= \sum_{k=0}^{6}(-1)^k\binom{6}{k}(6-k)! \\ &= \binom{6}{0}6! - \binom{6}{1}5! + \binom{6}{2}4! - \binom{6}{3}3! + \binom{6}{4}2! - \binom{6}{5}1! + \binom{6}{6}0! \\ &= 1\cdot 720 - 6\cdot 120 + 15\cdot 24 - 30\cdot 6 + 15\cdot 2 - 6\cdot 1 + 1\cdot 1 \\ &= 265. \end{aligned}$$

10. Note that $1800 = 25 \cdot 9 \cdot 8$. Use this information and the inclusion-exclusion formula to determine $\phi(1800)$, where ϕ is the Euler ϕ-function studied in class.

The prime factors of 1800 are 2, 3 and 5. So

$$\begin{aligned}\phi(1800) &= 1800(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) \\ &= 1800 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot 45 \\ &= \frac{2400}{5} \\ &= 480.\end{aligned}$$

11. True–False. Mark in the left margin.

F 1. There is a graph on 928 vertices in which no two vertices have the same degree.

F 2. There is a poset with 7403 points having width 65 and height 98.

T 3. There is a poset with 7403 points having width 85 and height 98.

F 4. The permutation $(8, 1, 4, 9, 3, 6, 2, 7, 5)$ is a derangement.

F 5. The number of partitions of an integer n into even parts is the same as the number of partitions of n into parts that are all the same.

Fun! 6. The partitions of a deranged surjection can be effectively computed using inclusion-exclusion and the process will consistently result in a maximum antichain of prime factors.