Solutions

Student Name and ID Number

MATH 3012 Quiz 1, September 18, 2014, WTT

1. Consider the 26-element set consisting of the capital letters of the English alphabet: $\{A, B, C, \ldots, Z\}$.

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4×2

a. How many strings of length 12 can be formed if repetition of symbols is permitted?

26^{12} 26 choices for each of 12 places

b. How many strings of length 12 can be formed if repetition of symbols is *not* permitted?

$P(26,12)$ $26 \times 25 \times 24 \times \ldots \times 15$ keep losing choices

c. How many strings of length 12 can be formed using exactly four X's, three Y's and five Z's?

$\binom{12}{4,3,5}$ choose places for each symbol

d. How many strings of length 12 can be formed using exactly four X's, three Y's and five Z's if the three $Y's$ are required to occur consecutively in the string?

$\binom{10}{4,1,5}$ treat 3 Y's as one character.

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2. How many lattice paths from $(0,0)$ to $(24,31)$ do *not* pass through $(15,19)$?

$\binom{55}{24} - \binom{34}{15}\binom{21}{9}$ total − bad = good

12

6×2

3. How many integer valued solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 = 42, \quad x_1, x_2, x_3 > 0.$

$\binom{41}{2}$ 41 gaps, choose 2

b. $x_1 + x_2 + x_3 = 42, \quad x_1, x_2, x_3 \geq 0.$

$\binom{44}{2}$ 3 artificial; 44 gaps, choose 2

c. $x_1 + x_2 + x_3 < 42, \quad x_1, x_2, x_3 > 0.$

$\binom{41}{3}$ slack is positive

d. $x_1 + x_2 + x_3 \leq 42, \quad x_1, x_2, x_3 \geq 0.$

$\binom{45}{3}$ slack is non-negative

e. $x_1 + x_2 + x_3 = 42, \quad x_1, x_3 > 0, x_2 \geq 7.$

$\binom{35}{2}$ pre-assign 6

f. $x_1 + x_2 + x_3 = 42, \quad x_1, x_3 > 0, 0 < x_2 \leq 6.$

$\binom{41}{2} - \binom{35}{2}$ part a − part e

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4. Use the Euclidean algorithm to find $d = \gcd(420, 245)$.

$$245\overline{)420}^{\,1} \quad 420 - 245 = 175$$

$$175\overline{)245}^{\,1} \quad 245 - 175 = 70$$

$$70\overline{)175}^{\,2} \quad 175 - 140 = 35$$

$$35\overline{)70}^{\,2} \quad 70 - 70 = 0$$

$$\boxed{g.c.d(420, 245) = 35}$$

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5. Use your work in the preceding problem to find integers a and b so that $d = 420a + 245b$.

$420 = 1 \cdot 245 + 175$
$245 = 1 \cdot 175 + 70$
$175 = 2 \cdot 70 + 35$

$175 = 1 \cdot 420 - 1 \cdot 245$
$70 = 1 \cdot 245 - 1 \cdot 175$
$35 = 1 \cdot 175 - 2 \cdot 70$

$$\begin{aligned}35 &= 1 \cdot 175 - 2 \cdot 70 \\ &= 1 \cdot 175 - 2[1 \cdot 245 - 1 \cdot 175] \\ &= -2 \cdot 245 + 3 \cdot 175 \\ &= -2 \cdot 245 + 3[1 \cdot 420 - 1 \cdot 245] \\ &= 3 \cdot 420 - 5 \cdot 245\end{aligned}$$

$$\boxed{a = 3,\ b = -5}$$

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6. For a positive integer n, let s_n count the number of ternary strings of length n that do not contain 102 as a substring. Note that $s_1 = 3$, $s_2 = 9$ and $s_3 = 26$. Develop a recurrence relation for s_n and use it to compute s_4, s_5 and s_6.

______ 0 $\quad s_{n-1}$ all good

______ 1 $\quad s_{n-1}$ all good

______ 2 $\quad s_{n-1}$ but some are bad!

______ 1 0 2 $\quad s_{n-3}$. These are bad

so $s_n = 3s_{n-1} - s_{n-3}$

$s_4 = 3 \cdot s_3 - s_1 = 3 \cdot 26 - 3 = 78 - 3 = 75$

$s_5 = 3 \cdot s_4 - s_2 = 3 \cdot 75 - 9 = 225 - 9 = 216$

$s_6 = 3 \cdot s_5 - s_3 = 3 \cdot 216 - 26 = 648 - 26 = 622$

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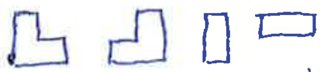
7. Let t_n denote the number of ways to tile a $2 \times n$ checkerboard using tiles of the two shapes shown on the white board. One of the shapes is an "L" with a total area of 3. As illustrated, this shape can be used "forwards" and "backwards" but not upside down. The other shape is a 2×1 strip and it can be used vertically or horizontally. Note that $t_1 = 1$, $t_2 = 2$ and $t_3 = 3$. Develop a recurrence for t_n and use it to compute t_4, t_5 and t_6.

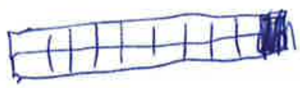

Allowable tiles

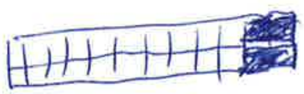

$t_1 = 1$

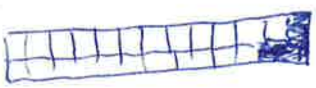

$t_2 = 2$


$t_3 = 3$

For recursion, look at how upper right corner is covered.

Case 1.  t_{n-1}

Case 2  t_{n-2}

Case 3  Now last part must be


t_{n-4}, t_{n-6}, t_{n-8}, ...

So $t_n = t_{n-1} + t_{n-2} + t_{n-4} + t_{n-6} + t_{n-8} + \cdots$

$t_4 = t_3 + t_2 + t_0 = 3 + 2 + 1 = 6$

$t_5 = t_4 + t_3 + t_1 = 6 + 3 + 1 = 10$

$t_6 = t_5 + t_4 + t_2 + t_0 = 10 + 6 + 2 + 1 = 19$

8. Find the coefficient of $x^4y^7z^{24}$ in $(6x - 5y + 8z^2)^{23}$

~~10~~ 8

$$6^4(-5)^7\,8^{12}\binom{23}{4,7,12}$$

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(12)
6×2

9. True–False. Mark in the left margin.

F 1. $P(7,3) = 1024$. $P(7,3) = 7 \cdot 6 \cdot 5 = 210$

T 2. $C(7,3) = 35$. $C(7,3) = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35$

T 3. If 57 pigeons are placed in 7 holes, then there is some hole with at least 9 pigeons. $7 \cdot 8 = 56$

T 4. If $f(n) = 865n + 90\log n$, and $g(n) = 3n + 7$, then $f(n) = O(g(n))$.

T 5. If $f(n) = 865n + 90\log n$, and $g(n) = 3n^2 + 7$, then $f(n) = o(g(n))$.

T 6. $\log n = o(\sqrt{n})$, $\sqrt{n} = o(n)$, $n = o(n^3)$, $n^3 = o(2^n)$, $2^n = o(2^{n^2})$ and $2^{n^2} = o(2^{2^n})$.

FUN! 7. A recursive permutation tiles non-distinct pigeons with a certificate that can be enumerated but not verified.

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All pages

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