Student Name and ID Number

MATH 3012, Quiz 3, November 24, 2015, WTT

1. Let n and m be positive integers with $n \geq m$.
a. Write the inclusion/exclusion formula for the number $S(n, m)$ of surjections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

b. Evaluate your answer in part a when $n = 6$ and $m = 4$.

2. Recall that $\phi(n)$ is the number of integers in $\{1, 2, \ldots, n\}$ which are relatively prime to n. Use inclusion/exclusion to evaluate $\phi(n)$ when $n = 2^3 \cdot 5^2 \cdot 11$.

3. Recall that $1/(1 - x)$ is the closed form for the generating function $1 + x + x^2 + x^3 + x^4 + \ldots$.
Find the closed form for the generating function: $1 - x/2 + x^2/3 - x^3/4 + x^4/5 - x^5/6 + - \ldots$

4. Write an expression (an infinite product) for the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_0 = 1$ and for $n \geq 1$, a_n is the number of partitions of the integer n into distinct parts, all of which are odd. For example $17 = 9 + 5 + 3$ and $44 = 23 + 11 + 9 + 1$.

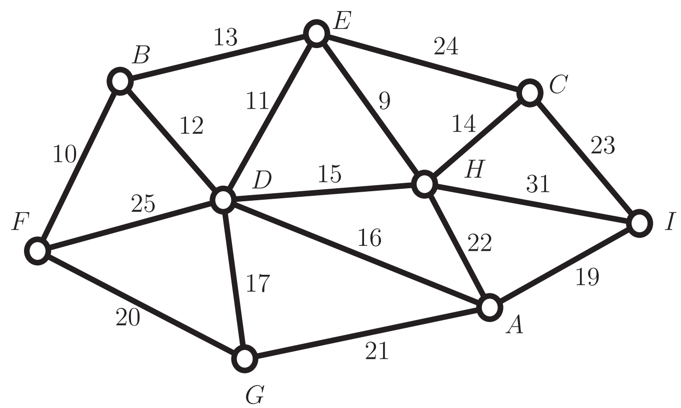
5. a. Find the general solution to the advancement operator equation:
$(A - 2 + i)^3 (A - 5)^2 f(n) = 0$

b. Find the general solution to the equation:
$(A^2 - 5A + 6) f(n) = 0.$

c. Find a particular solution to the equation:
$(A^2 - 5A + 6) f(n) = 14.$

d. Find the solution to the equation:
$(A^2 - 5A + 6) f(n) = 14$ subject to $f(0) = 6$ and $f(1) = 9$.

6. A graph with weights on edges is shown below. In the space to the right of the figure, list *in order* the edges which make up a minimum weight spanning tree using, respectively, Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.



Kruskal **Prim**

7. Dijkstra's algorithm is being run on a weighted digraph with vertex set $\{1, 2, \ldots, 8\}$ to find shortest paths from vertex 1 to all other vertices. After 5 iterations, the vertices marked *permanent* are $\{1, 3, 4, 7, 8\}$ and scans have been completed from each of these five vertices. Here are the shortest paths the algorithm has found thus far:

$P(1) = (1)$ total length 0.
$P(8) = (1, 8)$ total length 9.
$P(4) = (1, 4)$ total length 23.
$P(3) = (1, 8, 3)$ total length 24.
$P(6) = (1, 6)$ total length 28.

The candidate paths for the remaining three vertices are:

$P(2) = (1, 4, 2)$ total length 50.
$P(5) = (1, 8, 3, 5)$ total length 44.
$P(7) = (1, 6, 7)$ total length 82.

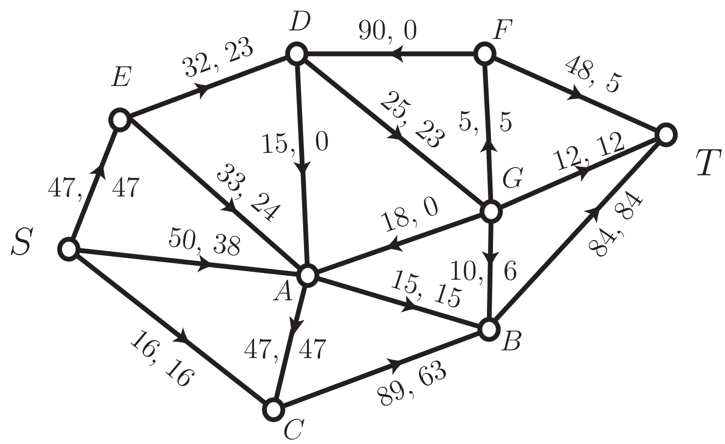
a. The weight $w(8, 3)$ of the edge $(8, 3)$ is __________.

b. The weight $w(6, 7)$ of the edge $(6, 7)$ is __________.

c. The temporary vertex which is now marked permanent is __________.

d. Which shortest paths $P(2)$, $P(5)$ and $P(7)$ will Dijkstra find if $w(2, 5) = w(2, 7) = 2$, $w(5, 2) = 4$, $w(5, 7) = 38$ and $w(7, 2) = 1$.

8. Consider the following network flow:



a. What is the current value of the flow?

b. What is the capacity of the cut $V = \{S, A, E, C\} \cup \{T, B, D, F, G\}$.

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices. Caution. The labelling should halt without the sink receiving a label.

d. Find a cut whose capacity is equal to the value of the current flow.

9. True–False. Mark in the left margin. Note: The first five of these questions are asked for the application of network flows (and bipartite matchings in particular) to solve the Dilworth problem for a poset P. In these five questions, the symbol G is used to represent the balanced bipartite graph associated with P.

1. When x and y are incomparable in P, the edge $x'y''$ is in G.
2. For every $x \in P$, the edge $x'x''$ is in G.
3. When the labelling algorithm halts and we obtain a maximum matching of size m in G, then we know that the width of P is m.
4. When $x'y''$ is an edge in the maximum matching, then x and y belong to distinct chains in the associated chain partition.
5. A maximum antichain in P can be obtained by selecting a point x from each chain in the chain partition associated with the maximum matching so that x' is labelled and x'' is unlabelled—when the labelling algorithm halts.
6. Let H be a bipartite graph with 250 vertices on one side and 400 on the other side. If the defect of H is 75, there is a matching of size 175 in H.
7. To implement Kruskal's algorithm, it is not necessary to sort the edges by weight. One can simply take the edges in any order and take the first one avoiding a cycle when added to those edges already chosen.
8. Dijkstra's algorithm finds shortest paths having the maximum number of edges.
9. The key idea behind the Ford-Fulkerson algorithm for network flows is to find at each step an augmenting path which maximizes the increase in the amount of the flow.
10. All linear programming problems posed with integral constraints have integral solutions.
11. Weakly convergent generating functions spanning Dilworth partitions admit Kruskal flows with irrational coefficients having distinct odd arrays.