Applications of the Probabilistic Method to Partially Ordered Sets

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This paper is dedicated to Paul Erdős with appreciation for his impact on mathematics and the lives of mathematicians all over the world.

Summary. There are two central themes to research involving applications of probabilistic methods to partially ordered sets. The first of these can be described as the study of random partially ordered sets. Among the specific models which have been studied are: random labelled posets; random t-dimensional posets; and the transitive closure of random graphs. A second theme concentrates on the adaptation of random methods so as to be applicable to general partially ordered sets. In this paper, we concentrate on the second theme. Among the topics we discuss are fibers and co-fibers; the dimension of subposets of the subset lattice; the dimension of posets of bounded degree; and fractional dimension. This last topic leads to a discussion of ramsey theoretic questions for probability spaces.

1. Introduction

Probabilistic methods have been used extensively throughout combinatorial mathematics, so it no great surprise to see that researchers have applied these techniques with great success to finite partially ordered sets. One central theme to this research is to define appropriate definitions of a *random poset*, and G. Brightwell's excellent survey article [1] provides a summary of work in this direction.

A second theme involves the application of random methods to more general classes of posets. After this brief introductory section, we present four examples of this theme. The first example is quite elementary and involves fibers and co-fibers, concepts which generalize the notions of chains and antichains. The prinicpal result here is an application of random methods to provide a non-trivial upper bound on the minimum size of fibers.

Our second example is more substantial. It involves the dimension of subposets of the subset lattice, an instance in which many of the classic techniques and results pioneered by Paul Erdős play major roles. The third example involves an application of the Lovász Local Lemma and leads naturally to the the investigation of the dimension of a random poset of height two.

Our last example involves fractional dimension for posets—an area where there are many attractive open problems. This topic leads to natural questions involving ramsey theory for probability spaces.

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The remainder of this section i
notation necessary for the remainin
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call X the *ground set* of the poset
The notations $x \leq y$ in P, $y \geq x$ in
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throughout the discussion. We writ
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In the remainder of this article
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and Hasse diagrams. For additional
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good source of background informa
article [2].

2. Fibers and Co-Fibers

The classic theorem of Dilworth [4]
can be partitioned into n chains. Al
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have devoted considerable energy to
antichains. Here is one such exampl

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of $\mathrm{cof}(\mathbf{P})$ taken over all n–element p
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$\mathrm{cof}(n) \leq \lfloor n/2 \rfloor$. On the other hand,
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Dually, a subset $B \subseteq X$ is called
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The remainder of this section is a very brief condensation of key ideas and notation necessary for the remaining five sections. In this article, we consider a *partially ordered set* (or *poset*) $\mathbf{P} = (X, P)$ as a discrete structure consisting of a set X and a reflexive, antisymmetric and transitive binary relation P on X. We call X the *ground set* of the poset $\mathbf{P}$, and we refer to P as a *partial order* on X. The notations $x \le y$ in P, $y \ge x$ in P and $(x, y) \in P$ are used interchangeably, and the reference to the partial order P is often dropped when its definition is fixed throughout the discussion. We write $x < y$ in P and $y > x$ in P when $x \le y$ in P and $x \ne y$. When $x, y \in X$, $(x, y) \notin P$ and $(y, x) \notin P$, we say x and y are *incomparable* and write $x \| y$ in P.

Although we are concerned almost exclusively with *finite* posets, i.e., those posets with finite ground sets, we find it convenient to use the familiar notation $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote respectively the reals, rationals, integers and positive integers equipped with the usual orders. Note that these four infinite posets are *total* orders; in each case, any two distinct points are comparable. Total orders are also called *linear* orders, or *chains*. We use $\mathbf{n}$ to denote an n–element chain with the points labelled as $0 < 1 < \ldots < n - 1$.

A subset $A \subseteq X$ is called an *antichain* if no two distinct points in A are comparable. We also use $\mathbf{P} + \mathbf{Q}$ to denote the disjoint sum of $\mathbf{P}$ and $\mathbf{Q}$.

In the remainder of this article, we will assume that the reader is familiar with the basic concepts for partially ordered sets, including maximal and minimal elements, chains and antichains, sums and cartesian products, comparability graphs and Hasse diagrams. For additional background information on posets, the reader is referred to the author's monograph [23], the survey article [14] on dimension by Kelly and Trotter and the author's survey articles [21], [22], [25] and [26]. Another good source of background information on posets is Brightwell's general survey article [2].

2. Fibers and Co-Fibers

The classic theorem of Dilworth [4] asserts that a poset $\mathbf{P} = (X, P)$ of width n can be partitioned into n chains. Also, a poset of height h can be partitioned into h antichains. For graph theorists, these results can be translated into the simple statement that comparability graphs are perfect. Against this backdrop, researchers have devoted considerable energy to generalizations of the concepts of chains and antichains. Here is one such example.

Let $\mathbf{P} = (X, P)$ be a poset. Lonc and Rival [18] called a subset $A \subseteq X$ a *co-fiber* if it intersects every non-trivial maximal chain in $\mathbf{P}$. Let $\mathrm{cof}(\mathbf{P})$ denote the least m so that $\mathbf{P}$ has a co-fiber of cardinality m. Then let $\mathrm{cof}(n)$ denote the maximum value of $\mathrm{cof}(\mathbf{P})$ taken over all n–element posets. In any poset, the set A_1 consisting of all maximal elements which are not minimal elements and the set A_2 of all minimal elements which are not maximal are both co-fibers. As $A_1 \cap A_2 = \emptyset$, it follows that $\mathrm{cof}(n) \le \lfloor n/2 \rfloor$. On the other hand, the fact that $\mathrm{cof}(n) \ge \lfloor n/2 \rfloor$ is evidenced by a height 2 poset with $\lfloor n/2 \rfloor$ minimal elements each of which is less than all $\lceil n/2 \rceil$ maximal elements. So $\mathrm{cof}(n) = \lfloor n/2 \rfloor$ (this argument appears in [18]).

Dually, a subset $B \subseteq X$ is called a *fiber* if it intersects every non-trivial maximal antichain. Let $\mathrm{fib}(\mathbf{P})$ denote the least m so that $\mathbf{P}$ has a fiber of cardinality m. Then let $\mathrm{fib}(n)$ denote the maximum value of $\mathrm{fib}(\mathbf{P})$ taken over all n-element posets. Trivially, $\mathrm{fib}(n) \ge \lfloor n/2 \rfloor$, and Lonc and Rival asked whether equality holds.

In [6], Duffus, Sands, Sauer and Woodrow showed that if $\mathbf{P} = (X, P)$ is an n-element poset, then there exists a set $F \subseteq X$ which intersects every 2-element maximal antichain so that $|F| \leq \lfloor n/2 \rfloor$. However, B. Sands then constructed a 17-point poset in which the smallest fiber contains 9 points. This construction was generalized by R. Maltby [19] who proved that for every $\epsilon > 0$, there exist a n_0 so that for all $n > n_0$ there exists an n-element poset in which the smallest fiber has at least $(8/15 - \epsilon)n$ points.

From above, there is no elementary way to see that there exists a constant $\alpha > 0$ so that $\text{fib}(n) < (1 - \alpha)n$. However, this is an instance where random methods provided real insights into the truth. In the remainder of this paper, we use the notation $[n]$ to denote the n–element set $\{1, 2, \ldots, n\}$ (No order is implied on $[n]$, except for the natural order on positive integers).

Theorem 2.1. *Let $\mathbf{P} = (X, P)$ be a poset with $|X| = n$. Then X contains a fiber of cardinality at most $4n/5$. Consequently,* $\text{fib}(n) \leq 4n/5$.

Proof. Let $C \subsetneq X$ be a maximum chain. Then $X - C$ is a fiber. So we may assume that $|C| < n/5$. Label the points of C as $x_1 < x_2 < \ldots < x_t$, where $t = |C| < n/5$. Next we define two different partitions of $X - C$. First, for each $i \in [t]$, set $U_i = \{x \in X - C : i \text{ is the least integer for which } x \| x_i\}$. Then set $D_i = \{x \in X - C : i \text{ is the largest integer for which } x \| x_i\}$.

Then for each subset $S \subseteq [t-1]$, define

$$B(S) = C \cup (\cup\{D_i : i \in S\}) \cup (\cup\{U_{i+1} : i \notin S\})$$

Note that for each $i \in [t-1]$, the maximality of C implies that $D_i \cap U_{i+1} = \emptyset$.

Claim 1. For every subset $S \subseteq [t-1]$, $B(S)$ is a fiber.

Proof. Let $S \subseteq [t-1]$ and let A be a non-trivial maximal antichain. We show that $A \cap B(S) \neq \emptyset$. This intersection is nonempty if $A \cap C \neq \emptyset$, so we may assume that $A \cap C = \emptyset$. Now the fact that C is a maximal chain implies that every point of C is comparable with one or more points of A. However, no point of C can be greater than one point of A and less than another point of A. Also, x_1 can only be less than points in A, and x_t can only be greater than points in A. It follows that $t \geq 2$ and that there is an integer $i \in [t-1]$ and points $a, a' \in A$ for which $x_i < a$ in P and $x_{i+1} > a'$ in P. Clearly, $a' \in D_i$ and $a \in U_{i+1}$. If $i \in S$, then $D_i \subset B(S)$, and if $i \notin S$, then $U_{i+1} \subset B(S)$. In either case, we conclude that $A \cap B(S) \neq \emptyset$.

Claim 2. The expected cardinality of $B(S)$ with all subsets $S \subseteq [t-1]$ equally likely is $t + 3(n-t)/4$.

Proof. Note that $C \subseteq B(S)$, for all S. For each element $x \in X - C$, let i and j be the unique integers for which $x \in D_i$ and $x \in U_j$. Then $j \neq i + 1$. It follows that the probability that x belongs to $B(S)$ is exactly $3/4$.

To complete the proof of the theorem, we note that there is some $S \subseteq [t-1]$ for which the fiber $B(S)$ has at most $t + 3(n-t)/4$ points. However, $t < n/5$ implies that $t + 3(n-t)/4 < 4n/5$.

The preceding theorem remains an interesting (although admittedly elementary) illustration of applying random methods to *general* partially ordered sets. Characteristically, it shows that an n-point poset has a fiber containing at most $4n/5$ points without actually producing the fiber. Furthermore, this is also an instance in which the constant provided by random methods can be improved by another approach.

The following result is due to Duffus, Kierstead and Trotter [5].

Theorem 2.2. (Duffus, Kierstead a
$\mathcal{H}$ be the hypergraph of non-trivial
number of $\mathcal{H}$ is at most 3.

Theorem 2.2 shows that $\text{fib}(n) \leq 2$
3-coloring of the hypergraph $\mathcal{H}$ of n
of any two of $\{B_1, B_2, B_3\}$ is a fibe
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width.

Theorem 2.3. (Lonc) *Let* $\mathbf{P} = (X,$
$\mathbf{P}$ *has a fiber of cardinality at most*

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3. Dimension Theory

When $\mathbf{P} = (X, P)$ is a poset, a line
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of P is called a *realizer* of $\mathbf{P}$ when P
only if $x < y$ in L, for every $L \in \mathcal{R}$.
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It is useful to have a simple test
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definition. Let $\text{inc}(\mathbf{P}) = \text{inc}(X, P)$
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$(x, y) \in \text{inc}(X, P)$, there exist distin
L and $y > x$ in L'.
Here is a more useful test. Call

1. $x \| y$ in P;
2. $z < x$ in P implies $z < y$ in P,
3. $w > y$ in P implies $w > x$ in P

The set of all critical pairs of $\mathbf{P}$
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every critical pair (x, y), there is sor
order L on X *reverses* (x, y) if $x >$
minimum number of linear extensio
For each $n \geq 3$, let $\mathbf{S}_n$ denote
$a_1, a_2, \ldots, a_n$, n maximal elements
$j \neq i$. The poset $\mathbf{S}_n$ is called the
Note that $\dim(\mathbf{S}_n)$ is at most n, si
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For integers k, r and n with $1 \leq$
consisting all k–element and all r–e
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Theorem 2.2. (Duffus, Kierstead and Trotter) *Let* $\mathbf{P} = (X, P)$ *be a poset and let* $\mathcal{H}$ *be the hypergraph of non-trivial maximal antichains of* $\mathbf{P}$. *Then the chromatic number of* $\mathcal{H}$ *is at most* 3. □

Theorem 2.2 shows that $\text{fib}(n) \leq 2n/3$, since whenever $X = B_1 \cup B_2 \cup B_3$ is a 3-coloring of the hypergraph $\mathcal{H}$ of non-trivial maximal antichains, then the union of any two of $\{B_1, B_2, B_3\}$ is a fiber. Quite recently, Lonc [17] has obtained the following intersesting result providing a better upper bound for posets with small width.

Theorem 2.3. (Lonc) *Let* $\mathbf{P} = (X, P)$ *be a poset of width* 3 *and let* $|X| = n$. *Then* $\mathbf{P}$ *has a fiber of cardinality at most* $11n/18$. □

I am still tempted to assert that $\lim_{n\to\infty} \text{fib}(n)/n = 2/3$.

3. Dimension Theory

When $\mathbf{P} = (X, P)$ is a poset, a linear order L on X is called a *linear extension* of P when $x < y$ in L for all $x, y \in X$ with $x < y$ in P. A set $\mathcal{R}$ of linear extensions of P is called a *realizer* of $\mathbf{P}$ when $P = \cap\mathcal{R}$, i.e., for all x, y in X, $x < y$ in P if and only if $x < y$ in L, for every $L \in \mathcal{R}$. The minimum cardinality of a realizer of $\mathbf{P}$ is called the *dimension* of $\mathbf{P}$ and is denoted $\dim(\mathbf{P})$.

It is useful to have a simple test to determine whether a family of linear extensions of P is actually a realizer. The first such test is just a reformulation of the definition. Let $\text{inc}(\mathbf{P}) = \text{inc}(X, P)$ denote the set of all incomparable pairs in $\mathbf{P}$. Then a family $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every $(x, y) \in \text{inc}(X, P)$, there exist distinct linear extensions $L, L' \in \mathcal{R}$ so that $x > y$ in L and $y > x$ in L'.

Here is a more useful test. Call a pair $(x, y) \in X \times X$ a *critical pair* if:

1. $x \| y$ in P;
2. $z < x$ in P implies $z < y$ in P, for all $z \in X$; and
3. $w > y$ in P implies $w > x$ in P, for all $w \in X$.

The set of all critical pairs of $\mathbf{P}$ is denoted $\text{crit}(\mathbf{P})$ or $\text{crit}(X, P$. Then it is easy to see that a family $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every critical pair (x, y), there is some $L \in \mathcal{R}$ with $x > y$ in L. We say that a linear order L on X *reverses* (x, y) if $x > y$ in L. So the dimension of a poset is just the minimum number of linear extensions required to reverse all critical pairs.

For each $n \geq 3$, let $\mathbf{S}_n$ denote the height 2 poset with n minimal elements $a_1, a_2, \ldots, a_n$, n maximal elements $b_1, b_2, \ldots, b_n$ and $a_i < b_j$, for $i, j \in [n]$ and $j \neq i$. The poset $\mathbf{S}_n$ is called the standard example of an n–dimensional poset. Note that $\dim(\mathbf{S}_n)$ is at most n, since $\text{crit}(\mathbf{S}_n) = \{(a_i, b_i) : i \in [n]\}$ and n linear extensions suffice to reverse the n critical pairs in $\text{crit}(\mathbf{S}_n)$. On the other hand, $\dim(\mathbf{S}_n) \geq n$, since no linear extension can reverse more than one critical pair.

4. The Dimension of Subposets of the Subset Lattice

For integers k, r and n with $1 \leq k < r < n$, let $\mathbf{P}(k, r; n)$ denote the poset consisting all k–element and all r–element subsets of $\{1, 2, \ldots, n\}$ partially ordered by inclusion. For simplicity, we use $\dim(k, r; n)$ to denote the dimension of $\mathbf{P}(k, r; n)$.

Historically, most researchers have concentrated on the case $k = 1$. In a classic 1950 paper in dimension theory, Dushnik [7] gave an exact formula for $\dim(1, r; n)$, when $r \geq 2\sqrt{n}$.

Theorem 4.1. (Dushnik) *Let n, r and j be positive integers with $n \geq 4$ and $2\sqrt{n} - 2 \leq r < n - 1$. If j is the unique integer with $2 \leq j \leq \sqrt{n}$ for which*

$$\lfloor \frac{n - 2j + j^2}{j} \rfloor \leq k < \lfloor \frac{n - 2(j-1) + (j-1)^2}{j-1} \rfloor,$$

then $\dim(1, r; n) = n - j + 1$. □

No general formula for $\dim(1, r; n)$ is known when r is relatively small in comparison to n, although some surprisingly tight estimates have been found. Here is a very brief overview of this work, beginning with an elementary reformulation of the problem. When L is a linear order on X, $S \subset X$ and $x \in X - S$, we say $x > S$ in L when $x > s$ in L, for every $s \in S$.

Proposition 4.2. $\dim(1, r; n)$ *is the least t so that there exist t linear orders $L_1, L_2, \ldots, L_t$ of $[n]$ so that for every r-element subset $S \subset [n]$ and every $x \in [n] - S$, there is some $i \in [t]$ for which $x > S$ in L_i.* □

Spencer [20] used this proposition to estimate $\dim(1, 2; n)$. First, he noted that by the Erdős/Szekeres theorem, if $n > 2^{2^t}$ and $\mathcal{R}$ is any set of t linear orders on $[n]$, then there exists a 3–element set $\{x, y, z\} \subset [n]$ so that for all $L \in \mathcal{R}$, either $x < y < z$ in L or $x > y > z$ in L. Thus $\dim(1, 2; n) > t$ when $n > 2^{2^t}$. On the other hand, if $n \leq 2^{2^t}$, then there exists a family $\mathcal{R}$ of t linear orders on $[n]$ so that for every 3–element subset $S \subset [n]$ and every $x \in S$, there exists some $L \in \mathcal{R}$ so that either $x < S - \{x\}$ in L or $x > S - \{x\}$ in L. Then let $\mathcal{S}$ be the family of $2t$ linear orders on X determined by adding to $\mathcal{R}$ the duals of the linear orders in $\mathcal{R}$. Clearly, the $2t$ linear orders in $\mathcal{S}$ satisfy the requirements of Proposition 4.2 when $r = 2$, and we conclude:

Theorem 4.3. (Spencer) *For all $n \geq 4$,*

$$\lg \lg n < \dim(1, 2; n) \leq 2 \lg \lg n.$$

□

Spencer [20] then proceeded to determine a more accurate upper bound for $\dim(1, 2; n)$ using a technique applicable to larger values of r. Let t be a positive integer, and let $\mathcal{F}$ be a family of subsets of $[t]$. Then let r be an integer with $1 \leq r \leq t$. We say $\mathcal{F}$ is *r-scrambling* if $|\mathcal{F}| \geq r$ and for every sequence $(A_1, A_2, \ldots, A_r)$ of r distinct sets from $\mathcal{F}$ and for every subset $B \subseteq [r]$, there is an element $\alpha \in [t]$ so that $\alpha \in A_\beta$ if and only if $\beta \in B$. We let $M(r, t)$ denote the maximum size of a r-scrambling family of subsets of $[t]$. Spencer then applied the Erdős/Ko/Rado theorem to provide a precise answer for the size of $M(2, t)$.

Theorem 4.4. (Spencer) $M(2, t) = \binom{t-1}{\lfloor \frac{t-2}{2} \rfloor}$, *for all $t \geq 4$.* □

As a consequence, Spencer observed that

$$\lg \lg n < \dim(1, 2; n) \leq \lg \lg n + (\frac{1}{2} + o(1)) \lg \lg \lg n.$$

Almost 20 years later, Füredi, Hajnal, Rödl and Trotter [13] were able to show that the upper bound in this inequality is tight, i.e.,

$\dim(1, 2; n) = \lg$

For larger values of r, Spencer u
bound.

Theorem 4.5. (Spencer) *For every*
that $M(r, t) > c^t$.

Proof. Let p be a positive integer a
p whose elements are subsets of $[t]$.
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So at least one of these sequences is
$\binom{p}{r} 2^r (2^r - 1)^t 2^{(p-r)t} < 2^{pt}$. Clearly
$e^{\frac{1}{r 2^r}}$ is a constant larger than 1.

Here's how the concept of scramblin
$\dim(q, r; n)$.

Theorem 4.6. (Spencer) *If $p = M$*

Proof. Let $\mathcal{F}$ be an r-scrambling fa
where $p = M(r, t)$. Then set $n = 2$
For each $\alpha \in [t]$, define a linear ord
x and y be distinct integers from $[n]$
$x > y$ in L_α if either

1. $\alpha \in A_u$ and $u \in Q_x - Q_y$, or
2. $\alpha \notin A_u$ and $u \in Q_y - Q_x$.

It is not immediately clear why
it is easy to check that this is so. N
$x \in [n] - S$. We must show that x
let $u_y = \min((Q_x - Q_y) \cup (Q_y \cup Q_x$
Since $\mathcal{F}$ is a r-scrambling family of s
$\alpha \in A_{u_y}$ if and only if $u_y \in Q_x$. It
L_α.

By paying just a bit of attention
actually yield the following upper b

Theorem 4.7. (Spencer) *For all r*

Of course, this bound is only mean
n, but in this range, it is surprisin
recent result due to Kierstead.

Theorem 4.8. (Kierstead) *If* 2

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$32 \lg(r + 2 - \lg \lg$

We will return to the issue of estim

$$\dim(1,2;n) = \lg\lg n + (\frac{1}{2} + o(1))\lg\lg\lg n.$$

For larger values of r, Spencer used random methods to produce the following bound.

Theorem 4.5. (Spencer) *For every $r \geq 2$, there exists a constant $c = c_r > 1$ so that $M(r,t) > c^t$.*

Proof. Let p be a positive integer and consider the set of all sequences of length p whose elements are subsets of $[t]$. There are 2^{pt} such sequences. The number of such sequences which fail to be r-scrambling is easily seen to be at most

$$\binom{p}{r} 2^r (2^r - 1)^t 2^{(p-r)t}.$$

So at least one of these sequences is a r-scrambling family of subsets of $[t]$ provided $\binom{p}{r} 2^r (2^r-1)^t 2^{(p-r)t} < 2^{pt}$. Clearly this inequality holds for $p > c^t$ where $c = c_r \sim e^{\frac{1}{r2^r}}$ is a constant larger than 1.

Here's how the concept of scrambling families is used in provide upper bounds for $\dim(q,r;n)$.

Theorem 4.6. (Spencer) *If $p = M(r,t)$ and $n = 2^p$, then $\dim(1,r;n) \leq t$.*

Proof. Let $\mathcal{F}$ be an r-scrambling family of subsets of $[t]$, say $\mathcal{F} = \{A_1, A_2, \ldots, A_p\}$ where $p = M(r,t)$. Then set $n = 2^p$ and let $Q_1, Q_2, \ldots, Q_n$ be the subsets of $[p]$. For each $\alpha \in [t]$, define a linear order L_α on the set $[n]$ by the following rules. Let x and y be distinct integers from $[n]$ and let $u = \min((Q_x - Q_y) \cup (Q_y - Q_x))$. Set $x > y$ in L_α if either

1. $\alpha \in A_u$ and $u \in Q_x - Q_y$, or
2. $\alpha \notin A_u$ and $u \in Q_y - Q_x$.

It is not immediately clear why L_α is a linear order on $[n]$ for each $\alpha \in [t]$, but it is easy to check that this is so. Now let S be an r–element subset of $[n]$ and let $x \in [n] - S$. We must show that $x > S$ in L_α for some $\alpha \in [t]$. For each $y \in S$, let $u_y = \min((Q_x - Q_y) \cup (Q_y \cup Q_x))$ and then consider the family $\{A_{u_y} : y \in S\}$. Since $\mathcal{F}$ is a r-scrambling family of subsets of $[t]$, there exists some $\alpha \in [t]$ such that $\alpha \in A_{u_y}$ if and only if $u_y \in Q_x$. It follows from the definition of L_α that $x > S$ in L_α.

By paying just a bit of attention to constants, the preceding results of Spencer actually yield the following upper bound on $\dim(1,r;n$.

Theorem 4.7. (Spencer) *For all $r \geq 2$, $\dim(1,r;n) \leq (1+o(1))\frac{1}{\lg e} r2^r \lg\lg n$.* □

Of course, this bound is only meaningful if r is relatively small in comparison to n, but in this range, it is surprisingly tight. The following lower bound is a quite recent result due to Kierstead.

Theorem 4.8. (Kierstead) *If $2 \leq r \leq \lg\lg n - \lg\lg\lg n$, then*

$$\frac{(r+2-\lg\lg n + \lg\lg\lg n)^2 \lg n}{32\lg(r+2-\lg\lg n+\lg\lg\lg n)} \leq \dim(1,r;n).$$

□

We will return to the issue of estimating $\dim(1,r;n)$ in the next section.

5. The Dimension of Posets of Bounded Degree

Given a poset $\mathbf{P} = (X, P)$ and a point $x \in X$, define the *degree* of x in $\mathbf{P}$, denoted $\deg_{\mathbf{P}}(x)$, as the number of points in X which are comparable to x. This is just the degree of the vertex x in the associated comparability graph. Then define $\Delta(\mathbf{P})$ as the maximum degree of $\mathbf{P}$. Finally, define $\mathrm{Dim}(k)$ as the maximum dimension of a poset $\mathbf{P}$ with $\Delta(\mathbf{P}) \leq k$. Rödl and Trotter were the first to prove that $\mathrm{Dim}(k)$ is well defined. Their argument showed that $\mathrm{Dim}(k) \leq 2k^2 + 2$. It is now possible to present a very short argument for this result by first developing the following idea due to Füredi and Kahn [12].

For a poset $\mathbf{P} = (X, P)$ and a point $x \in X$, let $U(x) = \{y \in X : y > x \text{ in } P\}$ and let $U[x] = U(x) \cup \{x\}$. Dually, let $D(x) = \{y \in X : y < x \text{ in } P\}$ and $D[x] = D(x) \cup \{x\}$. The following proposition admits an elementary proof. In fact, something more can be said, and we will comment on this in the next section.

Proposition 5.1. (Füredi and Kahn) *Let $\mathbf{P} = (X, P)$ be a poset and let L be any linear order on X. Then there exist a linear extension L' of P so that if (x, y) is a critical pair and $x > D[y]$ in L, then $x > y$ in L', so that $x > D[y]$ in L'.* □

Theorem 5.2. (Rödl and Trotter) *If $\mathbf{P} = (X, P)$ is a poset with $\Delta(\mathbf{P}) \leq k$, then* $\dim(\mathbf{P}) \leq 2k^2 + 2$.

Proof. Define a graph $\mathbf{G} = (X, E)$ as follows. The vertex set X is the ground set of $\mathbf{P}$. The edge set E contains those two element subsets $\{x, y\}$ for which $U[x] \cap U[y] \neq \emptyset$. Clearly, the maximum degree of a vertex in $\mathbf{G}$ is at most k^2. Therefore, the chromatic number of $\mathbf{G}$ is at most k^2+1. Let $t = k^2+1$ and let $X = X_1 \cup X_2 \cup \ldots \cup X_t$ be a partition of X into subsets which are independent in $\mathbf{G}$. Then for each $i \in [t]$, let L_i be any linear order on X with $X_i > X - X_i$ in L_i. Finally, define L_{t+i} to be any linear order on X so that:

1. $X_i > X - X_i$ in L_{t+i}, and
2. The restriction of L_{t+i} to X_i is the dual of the restriction of L_i to X_i.

We claim that for every critical pair $(x, y) \in \mathrm{crit}(\mathbf{P})$, if $x \in X_i$, then either $x > D[y]$ in L_i or $x > D[y]$ in L_{t+i}. This claim follows easily from the observation that any two points of $D[y]$ are adjacent in $\mathbf{G}$ so that $|D[y] \cap X_i| \leq 1$.

Füredi and Kahn [12] made a dramatic improvement in the upper bound for in $\mathrm{Dim}(k)$ by applying the Lovász Local Lemma [9]. We sketch their argument which begins with an application of random methods to provide an upper bound for $\dim(1, r; n)$. In this sketch, we make no attempt to provide the best possible constants.

Theorem 5.3. (Füredi and Kahn) *Let r and n be integers with $1 < r < n$. If t is an integer such that*

$$n\binom{n-1}{r}\left(\frac{r}{r+1}\right)^t < 1, \tag{5.1}$$

then $\dim(1, r; n) \leq t$. *In particular,* $\dim(1, r; n) \leq r(r+1)\log(n/r)$.

Proof. Let t be an integer satisfying the inequality given in the statement of the theorem. Then let $\{L_i : i \in [t]\}$ be a sequence of t random linear orders on X. The expected number of pairs (x, S) where S is an r–element subset of $[n]$, $x \in [n] - S$ and there is no $i \in [t]$ for which $x > S$ in L_i is exactly what the left hand side of this inequality is calculating. It follows that this quantity is less than one, so the probability that there are no such pairs is positive. This shows that $\dim(1, r; n) \leq t$. The estimate $\dim(1, r; n) \leq r(r+1)\log(n/r)$ follows easily.

Theorem 5.4. (Füredi and Kahn)
then $\dim(\mathbf{P}) \leq 100k\log^2 k$, *i.e.,* Dim

Proof. The inequality $\dim(\mathbf{P}) \leq 100$
$k \leq 1000$, so we may assume that k
Using the Lovász Local Lemma, we
$Y_2 \ldots \cup Y_m$, with $|D[x] \cap Y_i| \leq r$, fo
and let $s = \dim(1, r; q)$. We construc
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Then for each $j \in [s]$, define $L_{i,j}$

1. $Y_i > X - Y_i$ in $L_{i,j}$ and
2. if $a < b$ in M_j, then $Y_{i,a} < Y_{i,b}$

. Finally, for each $j \in [s]$, define $L_{i,s}$

1. $Y_i > X - Y_i$ in $L_{i,s+j}$,
2. if $a < b$ in M_j, then $Y_{i,a} < Y_{i,b}$
3. if $a \in [q]$, then the restriction o
   $L_{i,j}$ to $Y_{i,a}$.

.

Next we claim that if (x, y) is
$j \in [2s]$ so that $x > D[y]$ in $L(i, j$
$D[y]$ are adjacent in $\mathbf{G}$ so at most
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Let $x \in Y_{i,j_0}$. Then there exists som
and $D[y] \cap Y_{i,j} \neq \emptyset$. It follows th
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Finally, we note that $s = \dim($
$100k\log^2 k$ as claimed.

There are two fundamentally impor
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provided the following lower bound

Theorem 5.5. (Kierstead) *If* lg l

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Bounded Degree

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by first developing the following idea

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$D(x) = \{y \in X : y < x$ in $P\}$ and
admits an elementary proof. In fact,
ment on this in the next section.

$= (X,P)$ *be a poset and let* L *be any*
xtension L' *of* P *so that if* (x,y) *is a*
L', *so that* $x > D[y]$ *in* L'. □

$X,P)$ *is a poset with* $\Delta(\mathbf{P}) \leq k$, *then*

The vertex set X is the ground set of
subsets $\{x,y\}$ for which $U[x] \cap U[y] \neq$
in $\mathbf{G}$ is at most k^2. Therefore, the
$= k^2+1$ and let $X = X_1 \cup X_2 \cup \ldots \cup X_t$
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is exactly what the left hand side of
this quantity is less than one, so the
tive. This shows that $\dim(1,r;n) \leq t$.
follows easily.

Theorem 5.4. (Füredi and Kahn) *If* $\mathbf{P} = (X,P)$ *is a poset for which* $\Delta(\mathbf{P}) \leq k$, *then* $\dim(\mathbf{P}) \leq 100k\log^2 k$, *i.e.,* $\text{Dim}(k) \leq 100k\log^2 k$.

Proof. The inequality $\dim(\mathbf{P}) \leq 100k\log^2 k$ follows from the preceding theorem if $k \leq 1000$, so we may assume that $k > 1000$. Set $m = \lceil k/\log k \rceil$ and $r = \lceil 9\log k \rceil$. Using the Lovász Local Lemma, we see that there exists a partition $X = Y_1 \cup Y_2 \ldots \cup Y_m$, with $|D[x] \cap Y_i| \leq r$, for every $x \in X$. Now fix $i \in [m]$, let $q = rk+1$ and let $s = \dim(1,r;q)$. We construct a family $\mathcal{R}_\rangle = \{\mathcal{L}_{\rangle,|} : | \in [\in\int]\}$ as follows.

Let $\mathbf{G}$ be the graph on X defined in the proof of Theorem 5.2. Then let $\mathbf{G}_i$ be the subgraph induced by Y_i. Now it is easy to see that any point of Y_i is adjacent to at most rk other points in Y_i in the graph $\mathbf{G}_i$. It follows that the chromatic number of $\mathbf{G}_i$ is at most $rk+1$. Let $Y_i = Y_{i,1} \cup \ldots \cup Y_{i,q}$ be a partition into subsets each of which is independent in $\mathbf{G}_i$. Then let $\mathcal{R} = \{\mathcal{M}_| : | \in [\int]\}$ be a family of linear orders of $[q]$ so that for every r–element subset $S \subset [q]$ and every $x \in [q] - S$, there is some $j \in [s]$ for which $x > S$ in M_j.

Then for each $j \in [s]$, define $L_{i,j}$ as any linear order for which:

1. $Y_i > X - Y_i$ in $L_{i,j}$ and
2. if $a < b$ in M_j, then $Y_{i,a} < Y_{i,b}$ in $L_{i,j}$.

. Finally, for each $j \in [s]$, define $L_{i,s+j}$ as any linear order for which:

1. $Y_i > X - Y_i$ in $L_{i,s+j}$,
2. if $a < b$ in M_j, then $Y_{i,a} < Y_{i,b}$ in $L_{i,j}$, and
3. if $a \in [q]$, then the restriction of $L_{i,s+j}$ to $Y_{i,a}$ is the dual of the restriction of $L_{i,j}$ to $Y_{i,a}$.

Next we claim that if (x,y) is a critical pair and $x \in Y_i$, then there is some $j \in [2s]$ so that $x > D[y]$ in $L(i,j)$. To see this observe that any two points in $D[y]$ are adjacent in $\mathbf{G}$ so at most r points in $D[y]$ belong to Y_i, and all points of $D[y] \cap Y_i$ belong to distinct subsets in the partition of Y_i into independent subsets. Let $x \in Y_{i,j_0}$. Then there exists some $j \in [s]$ so that $j_0 > j$ in M_j whenever $j \neq j_0$ and $D[y] \cap Y_{i,j} \neq \emptyset$. It follows that either $x > D[y]$ in $L(i,j)$ or $x > D[y]$ in $L(i,s+j)$.

Finally, we note that $s = \dim(1,r;q) \leq r(r+1)\log(q/r)$, so that $\dim(\mathbf{P}) \leq 100k\log^2 k$ as claimed.

There are two fundamentally important problems which leap out from the preceding inequality limiting the dimension of posets of bounded degree, beginning with the obvious question: Is the inequality $\text{Dim}(k) = O(k\log^2 k)$ best possible? However, the details of the proof also suggest that the inequality could be improved if one could provide a better upper bound than $\dim(1,\log k;k) = O(\log^3 k)$. Unfortunately, the second approach will not yield much as Kierstead [15] has recently provided the following lower bound.

Theorem 5.5. (Kierstead) *If* $\lg\lg n - \lg\lg\lg n \leq r \leq 2^{\lg^{1/2} n}$, *then*

$$\frac{(r+2-\lg\lg n+\lg\lg\lg n)^2\lg n}{32\lg(r+2-\lg\lg n+\lg\lg\lg n)} \leq \dim(1,r;n) \leq \frac{2k^2\lg^2 n}{\lg^2 k}. \qquad (5.2)$$

□

As a consequence, it follows that $\dim(1,\log k;k) = \Omega(\log^3 k/\log\log k)$. So the remaining challenge is to provide better lower bounds on $\text{Dim}(k)$. Random methods seem to be our best hope. Here is a sketch of the technique used by Erdős, Kierstead and Trotter [8] to show that $\text{Dim}(k) = \Omega(k\log k)$.

For a fixed positive integer n, consider a random poset $\mathbf{P}_n$ having n minimal elements $a_1, a_2, \ldots, a_n$ and n maximal elements $b_1, b_2, \ldots, b_n$. The order relation is defined by setting $a_i < b_j$ with probability $p = p(n)$; also, events corresponding to distinct min-max pairs are independent.

Erdős, Kierstead and Trotter then determine estimates for the expected value of the dimension of the resulting random poset. The arguments are far too complex to be conveniently summarized here, as they make non-trivial use of correlation inequalities. However, the following theorem summarizes the lower bounds obtained in [8].

Theorem 5.6. (Erdős, Kierstead and Trotter)

1. *For every $\epsilon > 0$, there exists $\delta > 0$ so that if*

$$\frac{\log^{1+\epsilon} n}{n} < p \leq \frac{1}{\log n},$$

then

$$\dim(\mathbf{P}) > \delta\, pn \log pn, \text{ for almost all } \mathbf{P}.$$

2. *For every $\epsilon > 0$, there exist $\delta, c > 0$ so that if*

$$\frac{1}{\log n} \leq p < 1 - n^{-1+\epsilon},$$

then

$$\dim(\mathbf{P}) > \max\{\delta n,\, n - \frac{cn}{p \log n}\}\ , \text{ for almost all } \mathbf{P}.$$

□

The following result is then an easy corollary.

Corollary 5.7. (Erdős, Kierstead and Trotter) *For every $\epsilon > 0$, there exists $\delta > 0$ so that if*

$$n^{-1+\epsilon} < p \leq \frac{1}{\log n},$$

then

$$\dim(\mathbf{P}) > \delta \Delta(\mathbf{P}) \log n, \text{ for almost all } \mathbf{P}.$$

□

Summarizing, we now know that

$$\Omega(k \log k) = D(k) = O(k \log^2 k). \tag{5.3}$$

It is the author's opinion that the upper bound is more likely to be correct and that the proof of this assertion will come from investigating the dimension of a slightly different model of random height 2 posets. For integers n and k with k large but much smaller than n, we consider a poset with n minimal points and n maximal points. However, the comparabilities come from taking k random matchings.

The techniques used by Erdős, Kierstead and Trotter in [8] break down when $p = o(\log n/n)$. But this is just the point at which we can no longer guarantee that the maximum degree is $O(pn)$.

6. Fractional Dimension a
Probability Spaces

In many instances, it is useful to con
combinatorial parameter, as in many
on the original problem. In [3], Bright
fractional dimension for posets. This
results, and many appealing questions
questions with immediate connection

Let $\mathbf{P} = (X, P)$ be a poset and le
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for each incomparable pair (x, y), th
reverse the pair (x, y), i.e., $|\{i : 1$
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for which there exists a k–fold realiz
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Note that the standard example
dimension n. Brightwell and Schei
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inequality could be improved to fdim
Trotter [10], and the argument yield
much the same flavor as Brooks' the

Theorem 6.1. (Felsner and Trotter
poset with $|D(x)| \leq k$, for all $x \in X$.
then fdim($\mathbf{P}$) $< k+1$ *unless one of t*
standard example of a poset of dime

We do not discuss the proof of this
a strengthening of Proposition 5.1,
dimension of the poset $\mathbf{P}(1, r; n)$ is r
small fractional dimension. However
dimension in terms of fractional dim

Theorem 6.2. *If* $\mathbf{P} = (X, P)$ *is a*
$\dim(\mathbf{P}) \leq (2 + o(1))q \log n$.

Proof. Let $\mathcal{F}$ be a multi-realizer of P
pair $(x, y) \in \text{crit}(\mathbf{P})$. Then take t to

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Then let $\{L_1, \ldots, L_t\}$ be a sequence
are equally likely to be chosen. Then
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and these lead to some challenging p
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dimension has produced a number

6. Fractional Dimension and Ramsey Theory for Probability Spaces

In many instances, it is useful to consider a fractional version of an integer valued combinatorial parameter, as in many cases, the resulting LP relaxation sheds light on the original problem. In [3], Brightwell and Scheinerman proposed to investigate fractional dimension for posets. This concept has already produced some interesting results, and many appealing questions have been raised. Here's a brief sketch of some questions with immediate connections to random methods.

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F} = \{\mathcal{M}_\infty, \ldots, \mathcal{M}_\sqcup\}$ be a multiset of linear extensions of P. Brightwell and Scheinerman [3] call $\mathcal{F}$ a *k–fold realizer* of P if for each incomparable pair (x, y), there are at least k linear extensions in $\mathcal{F}$ which reverse the pair (x, y), i.e., $|\{i : 1 \leq i \leq t, x > y \text{ in } M_i\}| \geq k$. The *fractional dimension* of $\mathbf{P}$, denoted by $\mathrm{fdim}(\mathbf{P})$, is then defined as the least real number $q \geq 1$ for which there exists a k–fold realizer $\mathcal{F} = \{M_1, \ldots, M_t\}$ of P so that $k/t \geq 1/q$ (it is easily verified that the least upper bound of such real numbers q is indeed attained and is a rational number). Using this terminology, the *dimension* of $\mathbf{P}$ is just the least t for which there exists a 1–fold realizer of P. It follows immediately that $\mathrm{fdim}(\mathbf{P}) \leq \dim(\mathbf{P})$, for every poset $\mathbf{P}$.

Note that the standard example of an n–dimensional poset also has fractional dimension n. Brightwell and Scheinerman [3] proved that if $\mathbf{P}$ is a poset and $|D(x)| \leq k$, for all $x \in X$, then $\mathrm{fdim}(\mathbf{P}) \leq k + 2$. They conjectured that this inequality could be improved to $\mathrm{fdim}(\mathbf{P}) \leq k + 1$. This was proved by Felsner and Trotter [10], and the argument yielded a much stronger conclusion, a result with much the same flavor as Brooks' theorem for graphs.

Theorem 6.1. (Felsner and Trotter) *Let k be a positive integer, and let $\mathbf{P}$ be any poset with $|D(x)| \leq k$, for all $x \in X$. Then $\mathrm{fdim}(\mathbf{P}) \leq k+1$. Furthermore, if $k \geq 2$, then $\mathrm{fdim}(\mathbf{P}) < k+1$ unless one of the components of $\mathbf{P}$ is isomorphic to $\mathbf{S}_{k+1}$, the standard example of a poset of dimension $k+1$.* □

We do not discuss the proof of this result here except to comment that it requires a strengthening of Proposition 5.1, and to note that it implies that the fractional dimension of the poset $\mathbf{P}(1, r; n)$ is $r+1$. Thus a poset can have large dimension and small fractional dimension. However, there is one elementary bound which limits dimension in terms of fractional dimension.

Theorem 6.2. *If $\mathbf{P} = (X, P)$ is a poset with $|X| = n$ and $\mathrm{fdim}(\mathbf{P}) = q$, then $\dim(\mathbf{P}) \leq (2 + o(1)) q \log n$.*

Proof. Let $\mathcal{F}$ be a multi-realizer of P so that $\mathrm{Prob}_{\mathcal{F}}[x > y] \geq 1/q$, for every critical pair $(x, y) \in \mathrm{crit}(\mathbf{P})$. Then take t to be any integer for which

$$n(n-1)(1 - 1/q)^t < 1.$$

Then let $\{L_1, \ldots, L_t\}$ be a sequence of length t in which the linear extensions in $\mathcal{F}$ are equally likely to be chosen. Then the expected number of critical pairs which are not reversed is less than one, so the probability that we have a realizer of cardinality t is positive.

Felsner and Trotter [10] derive several other inequalities for fractional dimension, and these lead to some challenging problems as to the relative tightness of inequalities similar to the one given in Theorem 6.1. However, the subject of fractional dimension has produced a number of challenging problems which are certain to require random methods in their solutions. Here is two such problems, one of which has recently been solved.

A poset $\mathbf{P} = (X, P)$ is called an *interval order* if there exists a family $\{[a_x, b_x] : x \in X\}$ of non-empty closed intervals of $\mathbb{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. Fishburn [11] showed that a poset is an interval order if and only if it does not contain $\mathbf{2} + \mathbf{2}$ as a subposet. The interval order $\mathbf{I}_n$ consisting of all intervals with integer endpoints from $\{1, 2, \ldots, n\}$ is called the *canonical interval order*.

Although posets of height 2 can have arbitrarily large dimension, this is not true for interval orders. For these posets, large height is a prerequisite for large dimension.

Theorem 6.3. (Füredi, Hajnal, Rödl and Trotter) *If* $\mathbf{P} = (X, P)$ *is an interval order of height n, then*

$$\dim(\mathbf{P}) \le \lg\lg n + (1/2 + o(1)) \lg\lg\lg n. \tag{6.1}$$

□

The inequality in the preceding theorem is best possible.

Theorem 6.4. (Füredi, Hajnal, Rödl and Trotter) *The dimension of the canonical interval order satisfies*

$$\dim(\mathbf{I}_n) = \lg\lg n + (1/2 + o(1)) \lg\lg\lg n. \tag{6.2}$$

□

Although interval orders may have large dimension, they have bounded fractional dimension. Brightwell and Scheinerman [3] proved that the dimension of any finite interval order is less than 4, and they conjectured that for every $\epsilon > 0$, there exists an interval order with dimension greater than $4 - \epsilon$. We believe that this conjecture is correct, but confess that our intuition is not really tested. For example, no interval order is known to have fractional dimension greater than 3.

Motivated by the preceding inequalities and the known bounds on the dimension and fractional dimension of interval orders and the posets $\mathbf{P}(1, r; n)$, Brightwell asked whether there exists a function $f : \mathbb{Q} \to \mathbb{R}$ so that if $\mathbf{P} = (X, P)$ is a poset with $|X| = n$ and $\mathrm{fdim}(\mathbf{P}) = q$, then $\dim(\mathbf{P}) \le f(q) \lg\lg n$. If such a function exists, then the family $P(1, r; n)$ shows that we would need to have $f(q) = \Omega(2^q)$.

But we will show that there is no such function. The argument requires some additional notation and terminology. Fix integers n and k with $1 \le k < n$. We call an ordered pair (A, B) of k-element sets a *(k, n)-shift pair* if there exists a $(k+1)$-element subset $C = \{i_1 < i_2 < \cdots < i_{k+1}\} \subseteq \{1, 2, \ldots, n\}$ so that $A = \{i_1, i_2, \ldots, i_k\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}\}$. We then define the *$(k, n)$-shift graph* $\mathbf{S}(k, n)$ as the graph whose vertex set consists of all k-element subsets of $\{1, 2, \ldots, n\}$ with a k-element set A adjacent to a k-element set B exactly when (A, B) is a (k, n)-shift pair. Note that the $(1, n)$ shift graph $\mathbf{S}(1, n)$ is just a complete graph. It is customary to call a $(2, n)$-shift graph just a shift graph; similarly, a $(3, n)$-shift graph is called a *double shift graph*. The formula for the chromatic number of the $(2, n)$-shift graph $\mathbf{S}(2, n)$ is a folklore result of graph theory: $\chi(\mathbf{S}(2, n)) = \lceil \lg n \rceil$. Several researchers in graph theory have told me that this result is due to Andras Hajnal, but Andras says that it is not. In any case, it is an easy exercise.

The following construction exploits the properties of the shift graph to provide a negative answer for Brightwell's question.

Theorem 6.5. *For every* $m \ge 3$*, there exists a poset* $\mathbf{P} = (X, P)$ *so that*

1. $|X| = m^2$;
2. $\dim(X, P) \ge \lg m$; *and*
3. $\mathrm{fdim}(X, P) \le 4$.

Proof. The poset $\mathbf{P} = (X, P)$ is c
$i, j \le m\}$, so that $|X| = m^2$. Th
$x(i, j_1) < x(i, j_2)$ in P, for each $i \in$
for each $i \in [m]$, $x(i_1, j_1) < x(i_2, j_2$

We now show that $\dim(X, P) \ge$
$j \le m$, $x(i, j - i) \| x(j, m)$. Let dim
realizer of P. For each i, j with $1 \le$
$\{1, 2, \ldots, t\}$ so that $x(i, j - i) > x(j$
of the $(2, m)$ shift graph $\mathbf{S}(1, m)$ u
$t \ge \chi(\mathbf{S}(2, m) = \lceil \lg m \rceil$. To see that
with $1 \le i < j < k \le m$, let $\phi(\{i,$
$x(i, j - i) > x(j, m)$ in L_α and $x(j, k$
in P. However, since $(k - i) + (m -$
in P, so that $x(k, m) > x(i, j - i)$ i

$$x(i, j - i) > x(j, m) > x($$

The inequalities in equation 6.3 car
is a proper coloring of the shift gra
$\dim(X, P) \ge \lceil \lg m \rceil$.

Finally, we show that $\mathrm{fdim}(X,$
be the natural projection maps defi
Next, we claim that for each subse
of P so that $x > y$ in $L(A)$ if:

1. $x \| y$ in P;
2. $p_1(x) \in A$ and $p_1(y) \notin A$.

To show that such linear extens
Chapter 2 of [23]). Let $A \subseteq [m]$,
$p_1(x) \in A$ and $p_1(y) \notin A\}$. Now
is an alternating cycle of length p,
(subscripts are interpreted cyclicall
A. It follows that $u_k < v_{k+1}$ in P, f
$p_2(v_{k+1}) - p_2(u_k) > m$. Also, we kn
Thus $p_1(v_{k+1}) + p_2(v_{k+1}) > p_1(v_k)$
for all $k \in [p]$. The contradiction s
cycles. Thus the desired linear ext

Finally, we note that if we tak
$x > y$ in at least $s/4$ of the linear
observe that there are exactly $2^s/$
contain $p_1(y)$. This shows that fd
proof of the theorem.

Now we turn our attention to the
subset $D \subseteq X$ is called a *down s*
always imply that $x \in P$. The foll
known to other researchers in the

Theorem 6.6. *Let n be a positive*
ble shift graph $\mathbf{S}(3, n)$ *is the least*
subset lattice $\mathbf{2}^t$.

1. $|X| = m^2$;
2. $\dim(X, P) \geq \lg m$; *and*
3. $\text{fdim}(X, P) \leq 4$.

Proof. The poset $\mathbf{P} = (X, P)$ is constructed as follows. Set $X = \{x(i,j) : 1 \leq i, j \leq m\}$, so that $|X| = m^2$. The partial order P is defined by first defining $x(i, j_1) < x(i, j_2)$ in P, for each $i \in [m]$ whenever $1 \leq j_1 < j_2 \leq m$. Furthermore, for each $i \in [m]$, $x(i_1, j_1) < x(i_2, j_2)$ in P if and only if $(i_2 - i_1) + (j_2 - j_1) > m$.

We now show that $\dim(X, P) \geq \lg m$. Note first that for each i, j with $1 \leq i < j \leq m$, $x(i, j-i) \| x(j, m)$. Let $\dim(X, P) = t$, and let $\mathcal{R} = \{\mathcal{L}_\infty, \mathcal{L}_\in, \ldots, \mathcal{L}_\sqcup\}$ be a realizer of P. For each i, j with $1 \leq i < j \leq m$, choose an integer $\phi(\{i, j\}) = \alpha \in \{1, 2, \ldots, t\}$ so that $x(i, j-i) > x(j, m)$ in L_α. We claim that ϕ is a proper coloring of the $(2, m)$ shift graph $\mathbf{S}(1, m)$ using t colors, which requires that $\dim(X, P) = t \geq \chi(\mathbf{S}(2, m) = \lceil \lg m \rceil$. To see that ϕ is a proper coloring, let i, j and k be integers with $1 \leq i < j < k \leq m$, let $\phi(\{i, j\}) = \alpha$ and let $\phi(\{j, k\}) = \beta$. If $\alpha = \beta$, then $x(i, j-i) > x(j, m)$ in L_α and $x(j, k-j) > x(k, m)$ in L_α. Also, $x(j, m) > x(j, k-j)$ in P. However, since $(k - i) + (m - j + i) > m$, it follows that $x(k, m) > x(i, j - i)$ in P, so that $x(k, m) > x(i, j - i)$ in L_α. Thus,

$$x(i, j-i) > x(j, m) > x(j, k-j) > x(k, m) > x(i, j-i) \text{ in } P \qquad (6.3)$$

The inequalities in equation 6.3 cannot all be true. The contradiction shows that ϕ is a proper coloring of the shift graph $\mathbf{S}(2, m)$ as claimed. In turn, this shows that $\dim(X, P) \geq \lceil \lg m \rceil$.

Finally, we show that $\text{fdim}(X, P) \leq 4$. For each element $x \in X$, let p_1 and p_2 be the natural projection maps defined by $p(x) = i$ and $p_2(x) = j$ when $x = x(i, j)$. Next, we claim that for each subset $A \subset [m]$, there exists a linear extension $L(A)$ of P so that $x > y$ in $L(A)$ if:

1. $x \| y$ in P;
2. $p_1(x) \in A$ and $p_1(y) \notin A$.

To show that such linear extensions exist, we use the alternating cycle test (see Chapter 2 of [23]). Let $A \subseteq [m]$, and let $S(A) = \{(x, y) \in X \times X : x \| y$ in P, $p_1(x) \in A$ and $p_1(y) \notin A\}$. Now suppose that $\{(u_k, v_k) : 1 \leq k \leq p\} \subseteq S(A)$ is an alternating cycle of length p, i.e., $u_k \| v_k$ and $u_k \leq v_{k+1}$ in P, for all $k \in [p]$ (subscripts are interpreted cyclically). Let $k \in [m]$. Then $p_1(u_k) \in A$ and $p_1(v_{k+1}) \notin A$. It follows that $u_k < v_{k+1}$ in P, for each $k \in [p]$. It follows that $p_1(v_{k+1} - p_1(u_k) + p_2(v_{k+1}) - p_2(u_k) > m$. Also, we know that $m \geq p_1(v_k) - p_1(u_k) + p_2(v_k) - p_2(u_k)$. Thus $p_1(v_{k+1}) + p_2(v_{k+1}) > p_1(v_k) + p_2(v_k)$. Clearly, this last inequality cannot hold for all $k \in [p]$. The contradiction shows that $S(A)$ cannot contain any alternating cycles. Thus the desired linear extension $L(A)$ exists.

Finally, we note that if we take $\mathcal{F} = \{\mathcal{L}(\mathcal{A}) : \mathcal{A} \subseteq [\Updownarrow]\}$ and set $s = |\mathcal{F}|$, then $x > y$ in at least $s/4$ of the linear extensions in $\mathcal{F}$, whenever $x \| y$ in P. To see this, observe that there are exactly $2^s/4$ subsets of $[m]$ which contain $p_1(x)$ but do not contain $p_1(y)$. This shows that $\text{fdim}(X, P) \leq 4$ as claimed. It also completes the proof of the theorem.

Now we turn our attention to the double shift graph. If $\mathbf{P} = (X, P)$ is a poset, a subset $D \subseteq X$ is called a *down set*, or an *order ideal*, if $x \leq y$ in P and $y \in D$ always imply that $x \in P$. The following result appears in [13] but may have been known to other researchers in the area.

Theorem 6.6. *Let n be a positive integer. Then the chromatic number of the double shift graph $\mathbf{S}(3, n)$ is the least t so that there are at least n down sets in the subset lattice $\mathbf{2}^t$.* □

The problem of counting the number of down sets in the subset lattice $\mathbf{2}^t$ is a classic problem and is traditionally called Dedekind's problem. Although no closed form expression is known, relatively tight asymptotic formulas have been given. For our purposes, the estimate provided by Kleitman and Markovsky [16] suffices. Theorem 6.6, coupled with the estimates from [16] permit the following surprisingly accurate estimate on the chromatic number $\chi(\mathbf{S}(3,n))$ of the double shift graph [13].

Theorem 6.7. (Füredi, Hajnal, Rödl and Trotter)

$$\chi(\mathbf{S}(3,n)) = \lg\lg n + (1/2 + o(1)) \lg\lg\lg n.$$

□

Now that we have introduced the double shift graph, the following elementary observation can be made [13].

Proposition 6.8. *For each* $n \geq 3$, $\dim(1,2;n) \geq \chi(\mathbf{S}(3,n))$, *and* $\dim(\mathbf{I}_n) \geq \chi(\mathbf{S}(3,n))$. □

Although the original intent was to investigate questions involving the fractional dimension of posets, Trotter and Winkler [27] began to attack a ramsey theoretic problem for probability spaces which seems to have broader implications. Fix an integer $k \geq 1$, and let $n \geq k+1$. Now suppose that Ω is a probability space containing an event E_S for every k-element subset $S \subset \{1,2,\ldots,n\}$. We abuse terminology slightly and use the notation $\mathrm{Prob}(S)$ rather than $\mathrm{Prob}(E_S)$.

Now let $f(\Omega)$ denote the minimum value of $\mathrm{Prob}(A\overline{B})$, taken over all (k,n)-shift pairs (A,B). Note that we are evaluating the probability that A is true and B is false. Then let $f(n,k)$ denote the maximum value of $f(\Omega)$ and let $f(k)$ denote the limit of $f(n,k)$ as n tends to infinity.

Even the case $k=1$ is non-trivial, as it takes some work to show that $f(1) = 1/4$. However, there is a natural interpretation of this result. Given a sufficiently long sequence of events, it is inescapable that there are two events, A and B with A occurring before B in the sequence, so that

$$\mathrm{Prob}(A\overline{B}) < \frac{1}{4} + \epsilon.$$

The $\frac{1}{4}$ term in this inequality represents coin flips. The ϵ is present because, for finite n, we can always do slightly better than tossing a fair coin.

For $k=2$, Trotter and Winkler [27] show that $f(2) = 1/3$. Note that this is just the fractional chromatic number of the double shift graph. This result is also natural and comes from taking a random linear order L on $\{1,2,\ldots,n\}$ and then saying that a 2-element set $\{i,j\}$ is *true* if $i<j$ in L. Trotter and Winkler conjecture that $f(3) = 3/8$, $f(4) = 2/5$, and are able to prove that $\lim_{k\to\infty} f(k) = 1/2$. They originally conjectured that $f(k) = k/(2k+2)$, but they have since been able to show that $f(5) \geq \frac{27}{64}$ which is larger than $\frac{5}{12}$.

As an added bonus to this line of research, we are beginning to ask natural (and I suspect quite important) questions about patterns appearing in probability spaces.

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A Bound of the Car
Containing Δ-Systen

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Dedicated to Professor Paul Er

Summary. P.Erdős and R.Rado de
two members have the same intersect
maximum cardinality $\varphi(n)$ of an n–
of cardinality 3. Namely, we prove th
that for any n,

$$\varphi(n$$

1. Introduction

P. Erdős and R. Rado [2] introduced
$\mathcal{H}$ of finite sets a *Δ-system* if every t

Let $\varphi(n)$(respectively, $\gamma(n)$) den
uniform family (respectively, interse
Δ-system of cardinality 3.

P. Erdős and R. Rado [2] proved

$$2^n n$$

and conjectured that

$$\varphi(n) < c^n \quad \text{for}$$

The best published upper bound for

$$\varphi(n$$

Z. Füredi and J. Kahn (see [1]) prov

$$\varphi($$

The aim of the present paper is

Theorem 1.1. *For any integer $\alpha >$*

$$\varphi(n$$

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dation of Fundamental Researc
Science Foundation.