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The maximum number of edges in a graph of bounded dimension, with applications to ring theory

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Abstract

With a finite graph $G = (V, E)$, we associate a partially ordered set $\boldsymbol{P} = (X, P)$ with $X = V \cup E$ and $x < e$ in P if and only if x is an endpoint of e in $\boldsymbol{G}$. This poset is called the incidence poset of $\boldsymbol{G}$. In this paper, we consider the function $M(p,d)$ defined for $p, d \geqslant 2$ as the maximum number of edges a graph $\boldsymbol{G}$ can have when it has p vertices and the dimension of its incidence poset is at most d. It is easy to see that $M(p,2) = p - 1$ as only the subgraphs of paths have incidence posets with dimension at most 2. Also, a well known theorem of Schnyder asserts that a graph is planar if and only if its incidence poset has dimension at most 3. So $M(p,3) = 3p - 6$ for all $p \geqslant 3$. In this paper, we use the product ramsey theorem, Turán's theorem and the Erdős/Stone theorem to show that $\lim_{p\to\infty} M(p,4)/p^2 = 3/8$. We then derive some ring theoretic consequences of this in terms of minimal first syzygies and Betti numbers for monomial ideals.

Keywords: Graph; Partially ordered set; Dimension; Regularity lemma; Ramsey theory; Extremal graph theory

1. Introduction

In recent years, researchers have discovered interesting connections between graphs and the dimension of their incidence posets. Our goal here is to study the following extremal problem:

Problem 1.1. For integers $p, d \geqslant 2$, find the maximum number $M(p,d)$ of edges a graph on p vertices can have if the dimension of its incidence poset is at most d.

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The starting point for this research is the following well known theorem of Schnyder [12].

Theorem 1.2. *A graph G is planar if and only if the dimension of its incidence poset is at most* 3.

As an immediate consequence of Schnyder's theorem, we see that determining the value of $M(p,3)$ is just the same as finding the maximum number of edges in a planar graph on p vertices, so $M(p,3)=3p-6$ for all $p\geqslant 3$.

We can also determine the exact value of $M(p,2)$, as it is easy to see that the incidence poset of a graph has dimension at most 2 if and only if it is either a path or a subgraph of a path. It follows that $M(p,2)=p-1$, for all $p\geqslant 2$.

So in this paper, we concentrate on the determination of $M(p,4)$. In this case, we will will prove the following asymptotic formula.

Theorem 1.3.

$$\lim_{p\to\infty}\frac{M(p,4)}{p^2}=\frac{3}{8}.$$

The proof of our main theorem requires several powerful combinatorial tools, including the product ramsey theorem, Turán's theorem and the Erdős/Stone theorem. We shall also require some basic background material on dimension theory. For this, we refer the reader to Trotter's monograph [16].

The next section develops notation and terminology to help in applying these tools, while the third section contains the proofs of results necessary to verify Theorem 1.3. Section 4 discusses related combinatorial problems, and Section 5 presents some applications to ring theory, a topic which served as the original motivation for this line of research.

2. Combinatorial tools

Throughout the paper, we denote the n-element set $\{1,2,\ldots,n\}$ by $[n]$. Given a finite set S and an integer k with $0\leqslant k\leqslant |S|$, we denote the set of all k-element subsets of S by $\binom{S}{k}$. Given an integer t, finite sets $S_1,S_2,\ldots,S_t$ and an integer k, we call an element of $\binom{S_1}{k}\times\binom{S_2}{k}\times\cdots\times\binom{S_t}{k}$ a *grid* (also, a $\boldsymbol{k}^t$ grid).

The next theorem is the first of three powerful tools we need to prove our main theorem. We refer the reader to [8] for the proof and for additional material on applications of Ramsey theory.

Theorem 2.1 (The product Ramsey theorem). *Given positive integers m, k, r and t, there exists an integer n_0 so that if $S_1,S_2,\ldots,S_t$ are sets with $|S_i|\geqslant n_0$ for all $i\in[t]$,* *and f is any map which assigns to each $\boldsymbol{k}^t$ grid in $\binom{S_1}{k} \times \binom{S_2}{k} \times \cdots \times \binom{S_t}{k}$ a color from $[r]$, then there exist subsets $H_1, H_2, \ldots, H_t$ and a color $\alpha \in [r]$ so that*

1. *$H_i \in \binom{S_i}{m}$, for all $i = 1, 2, \ldots, t$, and*
2. *$f(g) = \alpha$ for every $\boldsymbol{k}^t$ grid $g \in \binom{H_1}{k} \times \binom{H_2}{k} \times \cdots \times \binom{H_t}{k}$.*

In what follows, we will refer to the least n_0 for which the conclusion of the preceding theorem holds as the product ramsey number $\mathrm{PRN}(m, k, r, t)$.

For integers p and t with $1 \leqslant t \leqslant p$, let $\boldsymbol{T}(p,t)$ denote the balanced complete t-partite graph on p vertices, i.e., if $p = qt + r$, where $0 \leqslant r < k$, then $\boldsymbol{T}(p,t)$ is a complete t-partite graph having $t - r$ parts of size q and r parts of size $q + 1$. Let $T(p,t)$ count the number of edges in $\boldsymbol{T}(p,t)$. Evidently,

$$T(p,t) = \binom{t-r}{2} q^2 + \binom{r}{2} (q+1)^2 + r(t-r)q(q+1).$$

The following well known theorem [14] is often viewed as the starting point for extremal graph theory.

Theorem 2.2 (Turán's theorem). *For positive integers p and t with $p \geqslant t + 1$, the maximum number of edges in a graph $\boldsymbol{G}$ on p vertices which does not contain a complete subgraph of size $t + 1$ is $T(p,t)$. Furthermore equality is obtained only when $\boldsymbol{G}$ is isomorphic to $\boldsymbol{T}(p,t)$.*

The asymptotic version of Turán's theorem is also of interest, as it serves to motivate material to follow.

Corollary 2.3. *For a positive integer t and a positive real number $\delta > 0$, there exists an integer p_0 so that if $p \geqslant p_0$ and $\boldsymbol{G}$ is a graph on p vertices having more than $(\frac{1}{2} - \frac{1}{2t} + \delta)p^2$ edges, then $\boldsymbol{G}$ contains a complete subgraph on $t + 1$ vertices.*

Given a graph $\boldsymbol{H}$ on n vertices and an integer $p \geqslant n$, let $T(\boldsymbol{H}, p)$ be the maximum number of edges a graph on p vertices can have if it does not contain $\boldsymbol{H}$ as a subgraph. So Turán's theorem is just the determination of $T(\boldsymbol{H}, p)$ in the special case where $\boldsymbol{H}$ is a complete graph.

Suppose that the chromatic number of $\boldsymbol{H}$ is $t + 1$. Then the complete balanced t-partite graph $\boldsymbol{T}(p,t)$ does not contain $\boldsymbol{H}$, as all its subgraphs are t-colorable. It follows that

$$\lim_{p \to \infty} \frac{T(\boldsymbol{H}, p)}{p^2} \geqslant \frac{1}{2} - \frac{1}{2t}.$$

The following classic theorem asserts that this is asymptotically the right answer — provided $t \geqslant 2$. The case where $t = 1$ is quite different, although this detail is not of concern in this paper.

Theorem 2.4 (The Erdős/Stone theorem). *Let H be a graph with chromatic number $t+1\geqslant 3$. Then*

$$\lim_{p\to\infty}\frac{T(H,p)}{p^2}=\frac{1}{2}-\frac{1}{2t}.$$

3. Proof of the principal theorem

In this section, we develop the results necessary for the proof of Theorem 1.3. We first present the lower bound. It is the easier of the two bounds.

Theorem 3.1. *If G is a graph whose chromatic number is at most* 4, *then the incidence poset of G has dimension at most* 4.

Proof. Let $V=V_1\cup V_2\cup V_3\cup V_4$ be a partition of the vertex set V of G into 4 independent sets. Then let L be any linear order on V. We denoted by L^d the dual of L, i.e., $x<y$ in L^d if and only if $x>y$ in L. Then define 4 linear orders L_1,L_2,L_3 and L_4 on V by the following rules:

1. In $L_1, L(V_1)<L(V_2)<L(V_3)<L(V_4)$;
2. In $L_2, L(V_4)<L(V_3)<L(V_2)<L(V_1)$;
3. In $L_3, L^d(V_3)<L^d(V_4)<L^d(V_1)<L^d(V_2)$;
4. In $L_4, L^d(V_2)<L^d(V_1)<L^d(V_4)<L^d(V_3)$.

Then extend these linear orders to linear extensions of the incidence poset of G by inserting the edges as 'low as possible'. It is just an easy exercise to show that this results in a realizer of the incidence poset so that it has dimension at most 4, as claimed. □

Oberving that a t-partite graph has chromatic number at most t, we can then write the following lower bound for $M(p,4)$.

Corollary 3.2. *For every $p\geqslant 4, M(p,4)\geqslant T(p,4)$.*

Examining the formula for the number of edges in $T(p,4)$, we have the following lower bound for $M(p,4)$.

Corollary 3.3.

$$\lim_{p\to\infty}\frac{M(p,4)}{p^2}\geqslant\frac{3}{8}.$$

Remark. Although asymptotically the same as $p\to\infty$, $T(p,4)$ is not equal to $M(p,4)$. In [2, Theorem 1.2] an explicit embedding of a graph on $p\geqslant 8$ vertices with $\frac{1}{2}(p^2+$

$5p-24)-(\lfloor\frac{1}{4}p\rfloor+1)(p-2\lfloor\frac{1}{4}p\rfloor)$ edges into $\mathbb{N}_0^4$ is given. Hence we have at least that $M(p,4)-T(p,4)\geqslant 2p-12$ for all $p\geqslant 8$.

Now we turn our attention to providing an upper bound on $M(p,4)$.

Lemma 3.4. *There exists an integer p_0 so that any graph $\boldsymbol{G}$ whose incidence poset has dimension at most 4 does not contain the balanced complete 5-partite graph $\boldsymbol{T}(5p_0,5)$.*

Proof. Set $m=2$, $k=1$, $r=(5!)^4$ and $t=5$. We show that the conclusion of the lemma holds for the value $p_0=\mathrm{PRN}(m,k,r,t)$.

To accomplish this, we assume that $\boldsymbol{G}$ is a graph so that

1. $\boldsymbol{G}$ contains a subgraph $\boldsymbol{H}$ isomorphic to $\boldsymbol{T}(5p_0,5)$, and
2. The incidence poset of $\boldsymbol{G}$ has dimension at most 4, as evidenced by the realizer $\mathscr{R}=\{L_1,L_2,L_3,L_4\}$.

We then argue to a contradiction.

Label the five disjoint independent sets in the copy of $\boldsymbol{T}(5p_0,5)$ as S_1, S_2, S_3, S_4 and S_5. We then define a coloring of the $\mathbf{1}^5$ grids in $\binom{S_1}{1}\times\binom{S_2}{1}\times\cdots\times\binom{S_5}{1}$ as follows. Each $\mathbf{1}^5$ grid is just a 5-element set containing one point from each S_i. Then consider the order of these 5 points in the four linear extensions L_1,L_2,L_3 and L_4. In each L_α, the 5 points can occur in any of 5! orders. So taking the 4 orders altogether, there are at $r=(5!)^4$ patterns.

Applying the product ramsey theorem, it follows that for each $i\in[5]$, there is a 2-element subset H_i contained in S_i so that all the grids these subsets produce receive the same color.

Now it follows easily that the linear orders treat the sets H_1,H_2,H_3,H_4 and H_5 as *blocks*, i.e., if a point from one block is over a point from another block in L_k, then both points from the first block are over both points from the second block in L_k.

In view of the preceding remarks, we can define 4 linear orders M_1,M_2,M_3 and M_4 on [5] by the rule $i<j$ in M_k if and only if both points from H_i are less than both points from H_j in L_k. Then set $\mathscr{S}=\{M_1,M_2,M_3,M_4\}$.

Claim 1. *For distinct $i,j,k\in[5]$, there is some $\alpha\in[4]$ so that i is larger than both j and k in M_α.*

To see that this claim is valid, consider a vertex $x\in H_i$ and an edge e with one end point in H_j and the other in H_k. Since x and e are incomparable in the incidence poset, there is some $\alpha\in[4]$ with $x>e$ in L_α. It follows that x is larger than both end points of e in L_α. In turn, this implies that i is larger than both j and k in M_α.

Claim 2. *For distinct $i,j\in[5]$, and for each vertex $x\in H_i$, there is a unique $\alpha\in[4]$ with $i>j$ in M_α and x the largest element of H_i in L_α.*

To see that this claim holds, note that it is enough to show that there is some $\alpha \in [4]$ with $i > j$ in M_α and x the largest element of H_i in L_α. The uniqueness of α follows from the symmetry of the parameters.

Now let y be the other vertex in H_i, and let z be any vertex from H_j. Then let e be the edge $\{y,z\}$. Since x and e are incomparable, there is some $\alpha \in [4]$ with $x > e$ in L_α. Since $x > e > z$ in L_α, $i > j$ in M_α. Since $x > e > y$ in L_α, x is the largest vertex in H_i in L_α.

This next claim follows immediately from Claim 2.

Claim 3. *For distinct $i,j \in [5]$, there are exactly two integers $\alpha,\beta \in [4]$ with $i > j$ in M_α and in M_β. Furthermore, the restrictions of L_α and L_β to H_i are dual.*

For distinct $i,j \in [5]$, let $\mathscr{S}(i > j) = \{\alpha,\beta\}$ be the 2-element set so that $i > j$ in M_α and in M_β.

Claim 4. *For distinct $i,j,k \in [5]$, $\mathscr{S}(i > j) \cap \mathscr{S}(i > k) \neq \emptyset$.*

To see that this claim is valid, suppose to the contrary that for distinct $i,j,k \in [5]$, $\mathscr{S}(i > j) \cap \mathscr{S}(i > k) = \emptyset$. After relabelling, we may assume that $i > j$ in M_1 and in M_2 while $i > k$ in M_3 and in M_4. It follows that $k > i > j$ in both M_1 and in M_2, while $j > i > k$ in both M_3 and in M_4. But this implies that there is no $\alpha \in [4]$ so that i is larger than both j and k in M_α, which contradicts Claim 1.

Claim 5. *For every $i \in [5]$, there exists an integer $\alpha(i) \in [4]$ so that $\alpha(i) \in \mathscr{S}(i > j)$, for all $j \in [5]$ with $i \neq j$.*

To see that this claim holds, note that if it fails, then by Claim 4 there are three distinct values $\alpha,\beta,\gamma \in [4]$ so that

$$\alpha \in \mathscr{S}(i > j) \cap \mathscr{S}(i > k),$$

$$\beta \in \mathscr{S}(i > k) \cap \mathscr{S}(i > l),$$

$$\gamma \in \mathscr{S}(i > l) \cap \mathscr{S}(i > j).$$

It follows that the restrictions of L_α, L_β and L_γ to H_i are identical, which contradicts Claim 3.

Now here is the contradiction which completes the argument. Observe that for each $i \in [5]$, i is the highest element of [5] in $M_{\alpha(i)}$. Clearly, this is impossible as there are only 4 orders in $\mathscr{S}$. $\square$

To complete the proof of Theorem 1.3, we need only appeal to the Erdős/Stone theorem. Let $\delta > 0$. Since the complete balanced 5-partite graph $\boldsymbol{T}(5p_0, 5)$ has chromatic number 5, it follows that if p is sufficiently large, then any graph $\boldsymbol{G}$ on p vertices with more than $(3/8+\delta)p^2$ edges contains $\boldsymbol{T}(5p_0,5)$ as a subgraph. Therefore the dimension of its incidence poset is at least 5.

4. Related results and directions

It may actually be possible to provide an exact formula for $M(p,4)$, at least when p is sufficiently large. The regularity forced by the Product Ramsey Theorem makes this a possibility.

For larger values of d, our results are not as precise. This is not surprising, because the problem is linked to the difficult combinatorial problem of determining the dimension of the complete graph. Researchers have studied this problem extensively, producing exact formulas for $p \leqslant 13$ and relative tight asymptotic estimates for large values of p. Following the notation in [16], we let $\dim(k,r;p)$ denote the dimension of the poset consisting of all k-element and r-element subsets of $\{1,2,\dots,p\}$ ordered by inclusion. Of course, finding the dimension of the complete graph on p vertices is then just the problem of determining $\dim(1,2;p)$. We refer the reader to Kierstead's forthcoming survey article [10] for additional details on this topic and an extensive bibliography.

Observe that if we know that $\dim(1,2;n+1) > d$, for integers n and d, then we may conclude that

$$\lim_{p\to\infty} \frac{M(p,d)}{p^2} \leqslant \frac{1}{2} - \frac{1}{2n}. \tag{1}$$

In [15], Trotter showed that $\dim(1,2;n) \leqslant 4$, when $p \leqslant 12$, and also that $\dim(1,2;13) = 5$. This would imply that

$$\lim_{p\to\infty} \frac{M(p,4)}{p^2} \leqslant \frac{1}{2} - \frac{1}{24},$$

so the asymptotic result we have proved for $M(p,4)$ in the preceding section is considerably stronger.

Unfortunately, for $d=5$, we know of no better bound than the one obtained from Eq. (1). In this same volume, Hoşten and Morris [9] derive a new formula for computing $\dim(1,2;n)$ and use this formula to show that $\dim(1,2;n) \leqslant 5$ if and only if $n \leqslant 81$. They also show that $\dim(1,2;n) \leqslant 6$ if and only if $n \leqslant 2646$ and $\dim(1,2;n) \leqslant 7$ if and only if $n \leqslant 1\,425\,464$. This discussion establishes the upper bound in the following result.

Theorem 4.1.

$$\frac{24}{50} \leqslant \lim_{p\to\infty} \frac{M(p,5)}{p^2} \leqslant \frac{40}{81}.$$

Proof. We sketch how the lower bound is derived. We show that the dimension of any graph with chromatic number at most 25 has dimension at most 5. In particular, the complete 25-partite graph has dimension at most 5, regardless of the part-sizes. The bound then follows from counting the number of edges in the balanced 25-partite graph.

To accomplish this, we group the 25 parts into 5 blocks, each with 5 subblocks. The 5 blocks are labelled $B_1, B_2, \ldots, B_5$. Then, within block B_i, we have 5 subblocks labelled $B_{i,1}, B_{i,2}, \ldots, B_{i,5}$. We consider the vertices themselves as positive integers. Within the subblocks, the order on vertices will either be in the natural order as integers, or the dual of this order. To distinguish between the two, whenever we write just $B_{i,j}$, we also imply that the order is just as it occurs in the set of integers. But when we write $B^*_{i,j}$, we mean that the subblock $B_{i,j}$ is to be in reverse order.

Now we describe 5 linear orders on the 5 blocks:

1. $B_4 < B_3 < B_2 < B_5 < B_1$ in L_1.
2. $B_5 < B_4 < B_3 < B_1 < B_2$ in L_2.
3. $B_1 < B_5 < B_4 < B_2 < B_3$ in L_3.
4. $B_2 < B_1 < B_5 < B_3 < B_4$ in L_4.
5. $B_3 < B_2 < B_1 < B_4 < B_5$ in L_5.

We pause to note that the construction thus far is cyclic, and it will remain so. To complete the construction, we describe the ordering of the 5 subblocks of B_i, for each $i = 1, 2, \ldots, 5$. In this description, our notation is cyclic.

1. $B_{i,1} < B_{i,2} < B_{i,3} < B_{i,4} < B_{i,5}$ in L_i.
2. $B^*_{i,5} < B^*_{i,4} < B^*_{i,3} < B^*_{i,2} < B^*_{i,1}$ in L_{i+1}.
3. $B^*_{i,1} < B^*_{i,3} < B^*_{i,5} < B^*_{i,2} < B^*_{i,4}$ in L_{i+2}.
4. $B^*_{i,1} < B^*_{i,4} < B^*_{i,2} < B^*_{i,5} < B^*_{i,3}$ in L_{i+3}.
5. $B_{i,5} < B_{i,4} < B_{i,3} < B_{i,2} < B_{i,1}$ in L_{i+4}.

To extend these linear orders to linear extensions of the incidence poset, we insert the edges as low as possible. Then to verify that we have constructed a realizer, it suffices to show that for every vertex x and every edge e not containing x as one of its endpoints, there is some $i \in [5]$ so that $x > e$ in L_i. Now let y and z denote the end points of e. So we must only show that there is some $i \in [5]$ with x over both y and z in L_i.

Taking advantage of the symmetry in the construction, we may assume that x belongs to block B_1.

If neither y nor z is in B_1, then x is over both y and z in L_1.

Now suppose that y also belongs to B_1 but that z does not. Then x is over y and z in L_1 unless $x < y$ in $\mathbb{Z}$. Now consider the case where $x < y$ in $\mathbb{Z}$. Because the restrictions of L_1 and L_2 to B_1 are dual, x is over y and z in L_2 unless z is in block B_2. So we also assume z is in B_2.

Now we observe that if x and y belong to the same subblock of B_1, then x is over y and z in L_4. On the other hand, if x and y are in different subblocks, the x is over y and z in L_5.

Next consider the case where x, y and z belong to B_1. Here there are two subcases. Suppose first that neither y nor z belong to the same subblock as x. If y and z belong to the same subblock, then x is over y and z in one of L_1 and L_2. If y and z belong to different subblocks, then we observe that subblocks $B_{1,1}$, $B_{1,3}$, $B_{1,4}$ and $B_{1,5}$ are the top subblocks in the 5 linear orders, so we may assume x is in subblock $B_{1,2}$. This subblock appears in second position in L_2 and in L_3, so it follows that we may assume that one of y and z is in subblock $B_{1,1}$ and the other is in $B_{1,4}$. Now observe that x is over y and z in L_4.

Note that since they are end points of an edge, we cannot have both y and z in the same subblock as x. So to complete the proof, we consider the case where y belongs to the same subblock as x, say $B_{1,i}$ but that z belongs to subblock $B_{1,j}$ with $i \neq j$. If $i<j$ in $\mathbb{Z}$, then x is over y and z in one of L_2 and L_5. So we may assume $i>j$. If $x>y$ in $\mathbb{Z}$, then x is over y and z in L_1. So we may assume $x<y$ in Z. Finally, note that with these conditions, x is over y and z in one of L_3 and L_4. □

As $d \to \infty$, the problem of determining $M(d,p)$ becomes essentially the same as finding the dimension of the complete graph K_p. The construction used by Spencer in [13] shows that if $d \geqslant 3$, and

$$p \leqslant 2^{\binom{d-1}{\lfloor \frac{d-1}{2} \rfloor}},$$

then the dimension of the complete graph K_p is at most d. Furthermore, it is an easy exercise to show that the dimension of any graph with chromatic number at most p is at most $d+2$.

On the other hand, in [7], Füredi, Hajnal, Rödl and Trotter show that $\dim(\boldsymbol{K}_p)$ is at least as large as the chromatic number of the double shift graph on $[p]$. This in turn is just the least d for which there are p antichains in the poset consisting of all subsets of $[d]$ ordered by inclusion, a fact that was used in the proof of the preceding theorem. Now the problem of estimating the number of antichains in the subset lattice is a well studied problem known as 'Dedekind's Problem'. Although no closed form answer is known, relatively good asymptotic results have been found (see [11], for example), and they suffice to show that

$$\dim(\boldsymbol{K}_p) \sim \lg\lg p + (1/2 + o(1))\lg\lg\lg p.$$

Inverting the preceding formula then allows us to give an asymptotic formula for $\lim_{p\to\infty} M(p,d)/p^2$ which is quite accurate when d is large.

5. Some algebraic applications

In this section we want to interpret the graph theoretical results from previous sections algebraically in terms of monomial ideals of the polynomial ring $k[X_1,\dots,X_d]$ where k is a field. We will in particular consider the case $d \leqslant 5$.

Let $R=k[X_1,\dots,X_d]$ and $I\subseteq R$ an ideal generated by $\{f_1,\dots,f_p\}\subseteq R$. These generators give rise to an R-module surjection ϕ: $R^p\to R$ given by

$$(r_1,\dots,r_p)\mapsto\sum_{i=1}^{p}r_if_i$$

and hence we have $I\cong R^p/\ker(\phi)$ as R-modules. The submodule $\ker(\phi)\subseteq R^p$ is called the *(first) syzygy module* of the p-tuple $(f_1,\dots,f_p)\in R^p$, and is denoted by Syz $(f_1,\dots,f_p)$. If now this module is generated by $\bar{r}_1,\dots,\bar{r}_q\in R^p$ as an R-module then every solution $\bar{x}=(x_1,\dots,x_p)$ to the linear equation

$$x_1f_1+\cdots+x_pf_p=0$$

is an R-linear combination of the $\bar{r}_1,\dots,\bar{r}_q$, that is, there are $h_1,\dots,h_q\in R$ such that $\bar{x}=\sum_{i=1}^{q}h_i\bar{r}_i$.

For a monomial ideal I of R whose minimal generators are the monomials $m_1,\dots m_p\in R$ let m_{ij} (and m_{ji}) denote the least common multiple, $\mathrm{lcm}(m_i,m_j)$, of m_i and m_j. If now $\{\bar{e}_i$: $i=1,\dots,p\}$ is the standard basis for R^p and $S_{ij}=(m_{ij}/m_i)\bar{e}_i-(m_{ij}/m_j)\bar{e}_j$ then $\mathrm{Syz}(m_1,\dots,m_p)$ is generated by

$$S=\{S_{ij}\colon\ 1\leqslant i<j\leqslant p\}\tag{2}$$

as an R-module. The S_{ij} are called the *minimal first syzygies* of the monomial ideal I. For short proof of this we refer to [1, p. 119] or [5, p. 322]. An analog result can be shown for $\mathrm{Syz}(g_1,\dots,g_p)$ where $\{g_1,\dots,g_p\}$ is an arbitrary Gröbner basis in R, see [4, p. 245].

Consider the elements S_{ij} from (2). If we have three distinct indices i,j and k such that m_k devides m_{ij}, then m_{ij} is divisible by all three monomials m_i,m_j and m_k, and hence also by m_{ik} and m_{kj}. Since

$$\frac{m_{ij}}{m_i}\bar{e}_i-\frac{m_{ij}}{m_j}\bar{e}_j=\frac{m_{ij}}{m_{ik}}\left(\frac{m_{ik}}{m_i}\bar{e}_i-\frac{m_{ik}}{m_k}\bar{e}_k\right)+\frac{m_{ij}}{m_{kj}}\left(\frac{m_{kj}}{m_k}\bar{e}_k-\frac{m_{kj}}{m_j}\bar{e}_j\right).$$

We get that

$$S_{ij}\in RS_{ik}+RS_{kj}\subseteq\sum_{(\alpha,\beta)\neq(i,j)}RS_{\alpha,\beta}.\tag{3}$$

Assume now on the contrary that $S_{ij}\in\sum_{(\alpha,\beta)\neq(i,j)}RS_{\alpha,\beta}$ for some $i<j$. By taking the projection down to the ith component we get an equation of the form

$$\frac{m_{ij}}{m_i}=\sum_{\beta>i,\beta\neq j}f_{i\beta}\frac{m_{i\beta}}{m_i}-\sum_{\alpha<i,\alpha\neq j}f_{\alpha i}\frac{m_{\alpha i}}{m_i},\tag{4}$$

where $f_{i\beta},f_{\alpha i}\in R$. Multiplying through by m_i and considering the coefficient of the monomial m_{ij} both sides of (4), we see that there must be a $\gamma\neq j$ such that $m_{\gamma i}$ divides m_{ij}, and hence m_γ divides m_{ij} and $\gamma\notin\{i,j\}$.

Assume from now on that our monomial ideal I, which is minimally generated by $m_1,\dots,m_p \in R$, is *generic*, that is, no variable X_l appears with the same nonzero exponent in two generators m_i and m_j of I. Let S be as in (2), we have then

Lemma 5.1. *For a generic monomial ideal I, minimally generated by $m_1,\dots,m_p$, there is a unique minimal subest M of S that generates* $\mathrm{Syz}(m_1,\dots,m_p)$ *as an R-module. M consists of all $S_{ij} \in S$ such that $m_k | m_{ij} \Leftrightarrow k \in \{i,j\}$.*

Proof. Let us first show that M generates $\mathrm{Syz}(m_1,\dots,m_p)$. It suffices to show that each S_{ij} is an R-linear combination of elements of M, that is $S_{ij} \in \mathrm{Span}_R(M)$: If not every $S_{ij} \in S$ is in $\mathrm{Span}_R(M)$, there is an S_{ij} not in $\mathrm{Span}_R(M)$ with the corresponding monomial m_{ij} minimal w.r.t. the partial order among monomials in R defined by divisibility. We have in particular that $S_{ij} \notin M$, and hence there is a $k \notin \{i,j\}$ such that $m_k | m_{ij}$. Hence we have that both m_{ik} and m_{kj} divide m_{ij} and since I is generic, neither m_{ik} nor m_{kj} is equal to m_{ij}. Therefore both S_{ik} and S_{kj} are in $\mathrm{Span}_R(M)$ by minimality of m_{ij}, and hence by (3) $S_{ij} \in \mathrm{Span}_R(M)$, a contradiction. Therefore $\mathrm{Span}_R(M) = \mathrm{Syz}(m_1,\dots,m_p)$.

Finally, the fact that M is a minimal subset of S generating $\mathrm{Syz}(m_1,\dots,m_p)$ and unique, is a simple consequence of the fact that no element $S_{ij} \in S$ is an R-linear combination of other elements in S, since (4) would imply $S_{ij} \notin M$. $\square$

If we for each $i \in \{1,\dots,p\}$ let $\bar{x}_i$ denote the point in $\mathbb{N}_0^d$ that corresponds to the monomial m_i (that is, $(a_1,\dots,a_d) \leftrightarrow X_1^{a_1} \cdots X_d^{a_d}$) then we see that the number of elements in M is simply the number of edges of the graph $G=(V,E)$ with vertex set $V=\{\bar{x}_1,\dots,\bar{x}_p\}$ and edgeset $E=\{\{\bar{x}_i,\bar{x}_j\}: \bar{x}_i \vee \bar{x}_j \geqslant \bar{x}_k \Leftrightarrow k \in \{i,j\}\}$ (here $\bar{a} \vee \bar{b}$ is the 'join' of $\bar{a},\bar{b} \in \mathbb{N}_0^d$, that is, the least element in $\mathbb{N}_0^d$ greater than or equal to both $\bar{a}$ and $\bar{b}$, w.r.t. the usual partial order of $\mathbb{N}_0^d$.) The number of edges in E is at most the maximal number of edges of a graph on p vertices of dimension d.

It is easy to see that an embedding of a graph G of dimension d into $\mathbb{N}_0^d$, can be done in a generic manner. Hence if $M(p,d)$ is as in Problem 1.1 then there is a generic monomial ideal I generated minimally by $m_1,\dots,m_p \in R = k[X_1,\dots,X_d]$ such that the set M form Lemma 5.1, has precisely $M(p,d)$ elements.

Hence we get the following corollary from Theorem 1.3:

Corollary 5.2. *If a is a real number $>3/8$ then there exists an integer p_a such that for any generic monomial ideal I generated minimally by $p>p_a$ monomials $m_1,\dots,m_p$ in 4 variables,* $\mathrm{Syz}(m_1,\dots,m_p)$ *can be generated by ap^2 minimal first syzygies. Moreover, 3/8 is the least real number with this property.*

Similarly one can write down a corollary of Theorem 4.1 about monomials in 5 variables instead of 4, by replacing the number '3/8' with '40/81' in Corollary 5.2. In that case, however, the least number playing the role of '40/81' is not 40/81 itself necessarily, but a real number in the closed interval [24/50, 40/81].

We have so far given 'down-to-earth' algebraic interpretations of the main results in previous sections in terms of generic monomial ideals and their minimal first syzygies. We will now explain briefly how further informations can be obtained from Theorem 1.3 in a more general setup, which is described thoroughly in [3, Section 3].

For a given generic monomial ideal $I \subseteq R$, the first syzygy-module is uniquely determined by the minimal elements of I, w.r.t. the partial order defined by divisibility, and hence can be denoted by $\mathrm{Syz}(I)$ or $\mathrm{Syz}(R/I)$ without any danger of ambiguity. Hence the minimal set M from Lemma 5.1 also depends solely on I or on the quotient R/I. The number of elements of M, $|M|$, turns out to be $\beta_2(R/I)$, the second Betti number of R/I:

Following the setup of Section 3 in [3], for any $W \subseteq \{1,2,\dots,p\}$ denote $\mathrm{lcm}\{m_i\colon i \in W\}$ by m_W. Let Δ_I be the *Scarf complex* of I, as the abstract simplicial complex on the set $\{1,2,\dots,p\}$ defined by

$$\Delta_I = \{U \subseteq \{1,2,\dots,p\}\colon m_U \neq m_W \text{ for all } W \neq U\}.$$

This complex is of dimension $d-1$, in the sense that the largest number of elments of a set U in Δ_I is $d-1$. Because of the geometric fact that Δ_I can be viewed as a subcomplex of the boundary complex of a polytope in $\mathbb{R}^d$, then each $U \in \Delta_I$ with $|U| = j \in \{1,\dots,d\}$ is called a $j-1$-face of Δ_I.

Now, for a graph $G=(V,E)$ of order dimension d there is an embedding $\theta\colon V \cup E \to [X_1,\dots,X_d]$ (the set of monomials of the polynomial ring $R=k[X_1,\dots,X_d]$) such that if $V=\{v_1,\dots,v_p\}$ and $E \subseteq \{\{v_i,v_j\}\colon 1 \leqslant i<j \leqslant p\}$ then θ satisfies

1. $\theta(\{v_i,v_j\}) = \mathrm{lcm}\{\theta(v_i),\theta(v_j)\}$
2. $\theta(v_k) | \theta(\{v_i,v_j\}) \Leftrightarrow k \in \{i,j\}$.

Let I_θ be the monomial ideal generated by $\{\theta(v_i)\colon i=1,\dots,p\}$. We see that the incidence poset of G is simply the poset induced by Δ_{I_θ} by considering the 0 and 1 faces of Δ_{I_θ} only. Hence to determine the maximal number of edges of a graph on p vertices of dimension d, is the same as determining the maximal number of 1-faces of a Scarf complex Δ_I among all monomial ideals I, which we can assume to be generic, in d variables minimally generated by p monomials. By Corollary 3.3 in [3] the number of j-faces of Δ_I is equal to the Betti number $\beta_{j+1}(R/I)$. Hence for general d our problem of detemining $M(p,d)$ is a special case of the Upper Bound Problem [3, p. 12]:

Problem 5.3. For $i \in \{1,\dots,d\}$ determine $\beta_i(d,p)$, the maximal Betti number among all ideals, minimally generated by p monomials in d variables.

It is easy to see that these maximal Betti numbers are attained among generic and *artinian* monomial ideals (that is, ideals I where R/I is finitely dimensional over the field k.) The Scarf complex of a generic artinian ideal I turns out to be the boundary of a simplical polytope, with one facet removed. Hence for a given generic artinian ideal I generated minimally by p monomials in d variables we have that for $j \in \{0,1,\dots,d-2\}$ the number f_j of j-faces of Δ_I, together with f_{d-1} which is the number of the $d-1$-

facets of $\Delta_I + 1$, satisfy the Dehn–Sommerville equations [17, p. 252]:

$$f_{j-1} = \sum_{i=j}^{d} (-1)^{d-i} \binom{i}{j} f_{i-1} \quad \text{for } 0 \leqslant j \leqslant d/2$$

which, in fact, is the complete list of all linear equations among $f_{-1}, f_0, \ldots, f_{d-1}$. Note that $f_{-1} = 1$ always, and $f_0 = p$, the number of generators of I. By definition of $\beta_i(d, p)$ we have therefore for a given monomial ideal I, that the numbers of faces of Δ_I satisfy

$$\begin{aligned} & f_0 = \beta_1(d, p) = p, \\ & f_j \leqslant \beta_{j+1}(d, p) \quad \text{for } j \in \{1, \ldots, d-2\}, \\ & f_{d-1} \leqslant \beta_d(d, p) + 1. \end{aligned}$$

Consider now the Dehn–Sommerville equations in the case $d = 4$:

$$\begin{aligned} & f_0 - f_1 + f_2 - f_3 = 0, \\ & f_2 - 2f_3 = 0. \end{aligned} \tag{5}$$

If now I is a generic artinian monomial ideal, minimally generated by p monomials and with f_1 maximal, that is $f_1 = M(p, 4)$, then we get from (5)

$$\begin{aligned} & f_0 = p, \\ & f_1 = M(p, 4), \\ & f_2 = 2M(p, 4) - 2p, \\ & f_3 = M(p, 4) - p, \end{aligned}$$

and hence we see that each f_j, where $j \in \{0, 1, 2, 3\}$, is maximal if $f_0 = p$ is fixed and f_1 is maximal. Thus we conclude that the maximal Betti numbers for monomial ideals in 4 variables satisfy

$$\begin{aligned} & \beta_1(4, p) = p, \qquad \beta_2(4, p) = M(p, 4), \\ & \beta_3(4, p) = 2M(p, 4) - 2p, \\ & \beta_4(4, p) = M(p, 4) - (p + 1). \end{aligned} \tag{6}$$

For functions $F, G : \mathbb{N} \to \mathbb{N}$ denote $\lim_{p \to \infty} F(p)/G(p) = 1$ by $F(p) \approx G(p)$. We have an asymptotic solution of Problem 5.3 from Theorem 1.3 in the case $d = 4$.

Corollary 5.4. *The maximal Betti numbers for a monomial ideal minimally generated by p monomials in 4 variables satisfy*

$$\begin{aligned} & \beta_1(4, p) = p, \qquad \beta_2(4, p) \approx 3p^2/8, \\ & \beta_3(4, p) \approx 3p^2/4, \qquad \beta_4(4, p) \approx 3p^2/8. \end{aligned}$$

Similarly the Dehn–Sommerville equations in the case $d=5$ are

$$\begin{aligned} &f_0-f_1+f_2-f_3+f_4=2,\\ &2f_1-3f_2+4f_3-5f_4=0,\\ &2f_1-3f_2+6f_3-10f_4=0 \end{aligned} \tag{7}$$

from which we can, in the same way as in the case $d=4$, deduce that

$$\begin{aligned} &\beta_1(5,p)=p, \qquad \beta_2(5,p)=M(p,5),\\ &\beta_3(5,p)=4M(p,5)-10p+20,\\ &\beta_4(5,p)=5M(p,5)-15p+30,\\ &\beta_5(5,p)=2M(p,5)-6p+11. \end{aligned} \tag{8}$$

From (6) and (8) we conclude

Corollary 5.5. *The Upper Bound Problem, Problem* 5.3, *is equivalent to Problem 1.1 for dimensions* $d\leqslant 5$.

Remark. The Dehn–Sommerville equations are $\lceil\frac{d}{2}\rceil$ linear equations relating $d+1$ unknowns $f_{-1},f_0,f_1,\dots,f_{d-1}$. Since $(d+1)-\lceil\frac{d}{2}\rceil=\lceil\frac{d+1}{2}\rceil\geqslant 4$ for $d\geqslant 6$, we see that it will be impossible to express each f_j, where $j\in\{-1,0,1,\dots,d-1\}$, as a linear combination of some fixed three f-variables f_α, f_β, and f_γ. Just as (6) and (8) were based on expressing each f_j as a linear combination of f_{-1}, f_0 and f_1, that can not be done for dimensions $d\geqslant 6$.

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