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Interval orders and dimension ☆

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Abstract

We show that for every interval order $\boldsymbol{X}$, there exists an integer t so that if $\boldsymbol{Y}$ is any interval order with dimension at least t, then $\boldsymbol{Y}$ contains a subposet isomorphic to $\boldsymbol{X}$. ⓒ 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

A partially ordered set (or poset) $\boldsymbol{X} = (X, P)$ is called an *interval order* if there is a function I assigning to each element $x \in X$ a closed interval $I(x) = [l_x, r_x]$ of the real line $\mathbb{R}$ so that for all $x, y \in X$, $x < y$ in P if and only if $r_x < l_y$ in $\mathbb{R}$. Recall that the *dimension* of a poset $\boldsymbol{X} = (X, P)$ is the least t so that the partial order P is the intersection of a family of t linear orders on X. For $n > 1$, a poset on $2n$ points may have dimension as large as n, but the dimension of an interval order on n points cannot be nearly as large. In fact, Füredi et al. [4] showed that the maximum dimension $d(n)$ of an interval order on n points satisfies:

$$d(n) = \lg\lg n + (\tfrac{1}{2} + o(1))\lg\lg\lg n. \tag{1}$$

Since the growth rate is quite slow, it is natural to ask what causes an interval order to have large dimension. Here we provide a somewhat surprising answer. We show that *all* small interval orders are contained in *any* interval order of sufficiently large dimension. More formally, we will prove the following theorem.

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Theorem 1.1. *For every interval order $\boldsymbol{X}$, there exists a postive integer $t = t(\boldsymbol{X})$ so that if $\boldsymbol{Y}$ is any interval order with dimension at least t, then $\boldsymbol{Y}$ contains a subposet isomorphic to $\boldsymbol{X}$.*

This theorem is quite special to the class of interval orders. For posets in general, no such result can hold. To see this, note that for each pair g, t of positive integers, there exists a height 2 poset $\boldsymbol{X}$ with dimension at least t and girth at least g. It is trivial to construct such posets from graphs with large girth and large chromatic number.

For additional background material on posets, we refer the reader to Trotter's monograph [7], and any terms not explicity defined here can be found be found in this source. Other good sources of background material include the survey articles [8,9].

2. Posets and dimension

Throughout this paper, we consider a poset $\boldsymbol{X} = (X, P)$ as a structure consisting of a set X (almost always finite) and a reflexive, antisymmetric and transitive binary relation P on X. We call X the *ground set* of the poset $\boldsymbol{X}$, and we call P a *partial order* on X. The notations $x \leqslant y$ in P, $y \geqslant x$ in P and $(x, y) \in P$ are used interchangeably, and the reference to the partial order P is often dropped when its definition is fixed throughout the discussion. We write $x < y$ in P and $y > x$ in P when $x \leqslant y$ in P and $x \neq y$. When $x, y \in X$, $(x, y) \notin P$ and $(y, x) \notin P$, we say x and y are incomparable and write $x \| y$ in P. When $\boldsymbol{X} = (X, P)$ is a poset, we call the partial order $P^d = \{(y, x)\colon (x, y) \in P\}$ the *dual* of P and we let $\boldsymbol{P}^d = (X, P^d)$.

When P is a binary relation on X and $Y \subseteq X$, we denote the *restriction* of P to Y by $P(Y)$. When P is a partial order on X, $Q = P(Y)$ is a partial order on Y and $\boldsymbol{Y} = (Y, Q)$ is called a *subposet* of $\boldsymbol{X} = (X, P)$. Also, we call $\boldsymbol{Y}$ the subposet *determined* by Y. More generally, whenever a poset $\boldsymbol{X} = (X, P)$ remains fixed in a discussion and $Y \subseteq X$, we let $\boldsymbol{Y}$ denote the subposet determined by Y.

For an integer $n \geqslant 1$, let $\boldsymbol{n}$ denote the n-element chain $0 < 1 < \cdots < n-1$. Also, when $\boldsymbol{X}$ and $\boldsymbol{Y}$ are posets, let $\boldsymbol{X} + \boldsymbol{Y}$ denote the disjoint sum of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

When P and Q are binary relations on a set X, we say Q is an *extension* of P when $P \subseteq Q$; a linear order L on X is called a *linear extension* of a partial order P on X when $P \subseteq L$. A family $\mathscr{R}$ of linear extensions of P is called a *realizer* of P (also, a realizer of $\boldsymbol{X}$) when $P = \bigcap \mathscr{R}$, i.e., for all x, y in X, $x < y$ in P if and only if $x < y$ in L, for every $L \in \mathscr{R}$. The minimum cardinality of a realizer of P is called the *dimension* of $\boldsymbol{X}$ and is denoted $\dim(\boldsymbol{X})$. Note that if $\boldsymbol{X}$ contains $\boldsymbol{Y}$, then $\dim(\boldsymbol{Y}) \leqslant \dim(\boldsymbol{X})$.

3. Interval assignments and representations

Let X be a finite set. A function I assigning to each $x \in X$ a closed (possibly degenerate) interval $I(x) = [l_x, r_x]$ of $\mathbb{R}$ is called an *interval assignment on* X. For each $x \in X$, l_x is called the *left*-end point of $I(x)$ and r_x is called the *right*-end point of $I(x)$. When the assignment I is fixed throughout a discussion, we call l_x and r_x the left- and right-end points of x.

Let $[a,b]$ and $[c,d]$ be closed intervals of $\mathbb{R}$; we write $[a,b] \triangleleft [c,d]$ when $b < c$ in $\mathbb{R}$. Whenever $\boldsymbol{X} = (X,P)$ is an interval order, an interval assignment I on X is called an *representation* of $\boldsymbol{X}$ when $x < y$ in P if and only if $I(x) \triangleleft I(y)$. For brevity, whenever I is an interval assignment on a set X and $x \in X$, we will use the alternate notation $[l_x, r_x]$ for the closed interval $I(x)$.

Note that we do not require that the end points of intervals in the range of an interval assignment be distinct. We even allow degenerate intervals, and I need not be an injection. On the other hand, an interval assignment I is said to be *distinguishing* if all intervals are non-degenerate and all end points are distinct. It is easy to see that every interval order has a distinguishing representation.

One of the most fundamental results for interval orders is the following forbidden subposet characterization theorem due to Fishburn [2].

Theorem 3.1. *A poset $\boldsymbol{X}$ is an interval order if and only if it does not contain a subposet isomorphic to* $\mathbf{2} + \mathbf{2}$.

A poset $\boldsymbol{X} = (X,P)$ is called a *weak order* if there exists a function $w : X \to \mathbb{R}$ so that $x < y$ in P if and only if $w(x) < w(y)$ in $\mathbb{R}$. Weak orders also admit a simple characterization by forbidden subposets (see [3], for example).

Proposition 3.2. *A poset $\boldsymbol{X} = (X,P)$ is a weak order if and only if it does not contain a subposet isormorphic to* $\mathbf{2} + \mathbf{1}$.

For additional background information on interval orders, the reader is encouraged to consult Fishburn's monograph [3] and Trotter's survey article [10].

4. Canonical interval orders and thickets

For an integer $n \geqslant 2$, let $\boldsymbol{I}_n = (I_n, P_n)$ denote the interval order determined by the non-degenerate intervals with integer end points from $\{1, 2, \ldots, n\}$. For this interval order, the identity map is a representation, although of course, not a distinguishing one. The interval orders in the family $\{\boldsymbol{I}_n \colon n \geqslant 2\}$ are called *canonical* interval orders. In [1], Bogart et al., showed that $\lim_{n \to \infty} \dim(\boldsymbol{I}_n) = \infty$. A much more precise estimate on the growth rate of $\dim(\boldsymbol{I}_n)$ was given by Füredi et al. [4], and this estimate is essential to the formula given in Eq. (1).

Theorem 4.1.

$$\dim(\boldsymbol{I}_n) = \lg \lg n + (\tfrac{1}{2} + o(1)) \lg \lg \lg n. \tag{2}$$

The following elementary result is immediate.

Proposition 4.2. *Let* $\boldsymbol{X}=(X,P)$ *be an interval order with* $|X|=n$. *Then* $\boldsymbol{X}$ *is isomorphic to a subposet of the canonical interval order* $\boldsymbol{I}_{2n}$.

Definition 4.3. An *n-thicket* is a poset $\boldsymbol{T}_n=(T_n,P_n)$ whose ground set T_n is the union $T_n=C_n\cup D_n$, where $C_n=\{x_i\colon 1\leqslant i\leqslant n\}$ and $D_n=\{y_{j,k}\colon 1\leqslant j<k\leqslant n\}$. Furthermore, we require that:

1. $x_i<x_{i+1}$ in P_n, for $i=1,2,\ldots,n-1$,
2. $x_i<y_{j,k}$ in P_n if and only if $1\leqslant i<j<k\leqslant n$, and
3. $y_{j,k}<x_i$ in P_n if and only if $1\leqslant j<k<i\leqslant n$.

It is important to recognize that the definition of a n-thicket $\boldsymbol{T}_n=(T_n,P_n)$ does not specify the order relation on T_n precisely. In particular, we do not know whether $y_{i,j}$ and $y_{k,l}$ are comparable or incomparable when $j\leqslant k\leqslant j+1$. Also, note that when $n=1$, $D_1=\emptyset$, and a 1-thicket is just a one element poset. The next lemma allows us to show that an interval order contains a canonical interval order $\boldsymbol{I}_n$ by showing that it contains a $3n$-thicket.

Lemma 4.4. *Let* $\boldsymbol{X}=(X,P)$ *be an interval order, and let* $n\geqslant 2$. *If* $\boldsymbol{X}$ *contains a subposet isomorphic to a* $3n$*-thicket, then* $\boldsymbol{X}$ *contains a subposet isomorphic to the canonical interval order* $\boldsymbol{I}_n$.

Proof. Suppose that $\boldsymbol{X}$ contains a subposet isormorphic to a $3n$-thicket. Label the points which form this subposet using the notation from Definition 4.3. Then define an injection $f:I_n\to X$ by setting $f([i,j])=y_{3i-1,3j}$. It is easy to see that f is an order preserving injection. □

5. Linear extensions of interval orders

Let I be a representation of an interval order $\boldsymbol{X}=(X,P)$. A *choice function* f on I is an injection $f:X\to\mathbb{R}$ such that $l_x\leqslant f(x)\leqslant r_x$ in $\mathbb{R}$, for all $x\in X$. For a choice function f, the f*-extension of* I is the linear order $L(f)$ obtained by setting $x<y$ in $L(f)$ if and only if $f(x)<f(y)$ in $\mathbb{R}$. We state the following elementary result for emphasis.

Proposition 5.1. *Let* I *be a representation of an interval order* $\boldsymbol{X}$. *For every choice function* f *on* I, $L(f)$ *is a linear extension of* P.

The notion of a choice function allows us to provide a very short proof of an important lemma due to Rabinovitch [6].

Lemma 5.2. *Let* $\boldsymbol{X}=(X,P)$ *be an interval order, and let* $X=Y\cup Z$ *be a partition of* X. *Then there exists a linear extension* L *of* P *such that* $y>z$ *in* L *whenever* $y\in Y$, $z\in Z$ *and* $y\|z$ *in* P.

Proof. Let I be a distinguishing representation of $\boldsymbol{X}$. Define a choice function f on I by setting $f(y)=r_y$, for all $y\in Y$ and $f(z)=l_z$, for all $z\in Z$. □

It is customary to say that Y is *over* Z in the linear extension L constructed in the preceding Lemma. This is a slight abuse of terminology, since we do not require that $y>z$ in L for all $y\in Y$ and all $z\in Z$. We only require this ordering to hold when y is incomparable to z in P.

The next lemma asserts that all linear extensions of interval orders arise from choice functions.

Lemma 5.3. *Let I be a distinguishing representation of an interval order $\boldsymbol{X}=(X,P)$. For every linear extension L of P, there exists a choice function f on I such that $L=L(f)$.*

Proof. We argue by induction on $|X|$. The base step $|X| = 1$ is trivial, so assume the lemma holds for every interval order whose ground set has smaller cardinality. Let x_0 be the L-largest element of X. Then x_0 is also a maximal element of $\boldsymbol{X}$. Set $Y=X-\{x_0\}$. Let L' and I' be the restrictions of L and I to Y, respectively. By the induction hypothesis, there exists a choice function f' on I' such that $L'=L(f')$.

Let $Z=\{y\in Y\colon f'(y)\geqslant r_{x_0}\}$. If $Z=\emptyset$, we extend f' to X by setting $f(x_0)=r_{x_0}$ and $f(y)=f'(y)$, for all $y\in Y$. In this case, it is obvious that $L=L(f)$. So we may suppose that $Z\neq\emptyset$. Now let $p=\max\{f'(y)\colon y\in Y-Z\}$. Then $p<r_{x_0}$. Let $s=\max\{l_y\colon y\in Z\}$. Since x_0 is maximal in $\boldsymbol{X}$, and I is distinguishing, $s<r_{x_0}$. Set $q=\max\{s,p\}$. Then $q<r_{x_0}$. Let g be an order preserving injection from $\{f'(z)\colon z\in Z\}$ into the open interval (q,r_{x_0}) of $\mathbb{R}$. Finally, define a choice function f on I as follows.

$$f(x)=\begin{cases} r_{x_0} & \text{if } x=x_0,\\ g\circ f'(z) & \text{if } z\in Z,\\ f'(y) & \text{if } y\in Y-Z.\end{cases}$$

Clearly $L=L(f)$. □

The next lemma is again very special to interval orders. It does not hold for posets in general.

Lemma 5.4. *Let $\boldsymbol{X}=(X,P)$ be an interval order, and let $X=X_1\cup X_2\cup\cdots\cup X_s$ be a partition. For each $i=1,2,\ldots,s$, let L_i be a linear extension of $P(X_i)$. Then there exists a linear extension L of P so that for each $i=1,2,\ldots,s$, $L_i=L(X_i)$.*

Proof. Let I be a distinguishing representation of $\boldsymbol{X}$. For each $i=1,2,\ldots,s$, let I_i be the restriction of I to X_i. By Lemma 5.3, for each $i=1,2,\ldots,s$, there exists a choice functions f_i on I_i so that $L_i=L(f_i)$. Clearly, we may assume that $f(x)\neq f(y)$, when $x\in X_i$, $y\in X_j$ and $1\leqslant i<j\leqslant s$. Then define a choice function f on I by setting $f(x)=f_i(x)$, when $x\in X_i$. Finally, set $L=L(f)$. □

Let I be a distinguishing representation of an interval order $\boldsymbol{X}=(X,P)$. Then define $\dim^*(\boldsymbol{X},I)$ as the least $t\geqslant 0$ for which there exists a family $\mathscr{R}$ of t linear extensions of P so that if x and y are distinct incomparable points of $\boldsymbol{X}$ and $l_x < l_y < r_x < r_y$, then there exists $L\in\mathscr{R}$ with $x > y$ in L. Strictly speaking, $\dim^*(\boldsymbol{X},I)$ depends only on I, but we use this notation as a reminder of the poset for which I is a representaton. As the next lemma shows, the value of $\dim^*(\boldsymbol{X},I)$ does not stray too far from $\dim(\boldsymbol{X})$.

Lemma 5.5. *Let I be a distinguishing representation of an interval order* $\boldsymbol{X}=(X,P)$. *Then*

$$\dim^*(\boldsymbol{X},I)\leqslant\dim(\boldsymbol{X})\leqslant\dim^*(\boldsymbol{X},I)+2. \tag{3}$$

Proof. The inequality $\dim^*(\boldsymbol{X},I)\leqslant\dim(\boldsymbol{X})$ is immediate. We now show that $\dim(\boldsymbol{X})\leqslant\dim^*(\boldsymbol{X},I)+2$.

For each $x\in X$, let $f(x)=l_x$ and $g(x)=r_x$. Then set $M_1=L(f)$ and $M_2=L(g)$. Now suppose that $\dim^*(\boldsymbol{X},I)=t$ as evidenced by the family $\mathscr{R}$ of t linear extensions of P. Then $\{M_1,M_2\}\cup\mathscr{R}$ is a realizer of P so that $\dim(\boldsymbol{X})\leqslant t+2$, as claimed. □

It is important to note that the value of $\dim^*(\boldsymbol{X},I)$ depends on the representation I. For example, consider a three-element interval order $\boldsymbol{X}=(X,P)$ with $X=\{x,y,z\}$ as defined by the distinguishing representation $I(x)=[1,3]$, $I(y)=[2,5]$ and $I(z)=[4,6]$. The poset $\boldsymbol{X}$ is isomorphic to $\mathbf{2}+\mathbf{1}$, and it is easy to see that $\dim^*(\boldsymbol{X},I)=2$. On the other hand, the function $J(x)=[2,3]$, $J(y)=[1,6]$ and $J(z)=[4,5]$ is also a distinguishing representation of $\boldsymbol{X}$ and $\dim^*(\boldsymbol{X},J)=0$.

When I is a distinguishing representation of an interval order $\boldsymbol{X}=(X,P)$, $Y\subseteq X$ and J is the restriction of I to Y, we write $\dim^*(\boldsymbol{Y},I)$ rather than $\dim^*(\boldsymbol{Y},J)$.

Lemma 5.6. *Let I be a distinguishing representation of an interval order* $\boldsymbol{X}=(X,P)$, *and let* $X=X_1\cup X_2$ *be a partition of X into two non-empty parts. Then*

1. $\dim(\boldsymbol{X})\leqslant 2+\max\{\dim(\boldsymbol{X}_1),\dim(\boldsymbol{X}_2)\}$, *and*
2. $\dim^*(\boldsymbol{X},I)\leqslant 2+\max\{\dim^*(\boldsymbol{X}_1,I),\dim^*(\boldsymbol{X}_2,I)\}$.

Proof. We prove Statement 1. The argument for Statement 2 is similar. Let $t=\max\{\dim(\boldsymbol{X}_1),\dim((\boldsymbol{X}_2)\}$. Then use Lemma 5.4 to choose a family $\mathscr{S}$ of t linear extensions of P so that for $i=1,2$, the restrictions of the linear extensions in $\mathscr{S}$ to X_i form a realizer of $\boldsymbol{X}_i$. Then set $\mathscr{R}=\mathscr{S}\cup\{M_1,M_2\}$, where X_1 is over X_2 in M_1 and X_2 is over X_1 in M_2. It is clear that $\mathscr{R}$ is a realizer of P. □

Lemma 5.7. *Let I be a distinguishing representation of an interval order* $\boldsymbol{X}=(X,P)$, *and let* $X=Y\cup Z$ *be a partition of X into two non-empty parts. Suppose further that Y is a subset of the maximal elements of* $\boldsymbol{X}$. *Then*

1. $\dim(\boldsymbol{X})\leqslant 1+\dim(\boldsymbol{Z})$, *and*
2. $\dim^*(\boldsymbol{X},I)\leqslant\max\{2,1+\dim^*(\boldsymbol{Z},I)\}$.

Proof. Again, we prove Statement 1 only. Suppose that $\dim(\boldsymbol{Z})=t$ as evidenced by a family $\mathscr{S}=\{M_1,M_2,\ldots,M_t\}$ of t linear extensions of $\boldsymbol{Z}$. Choose a linear extension L_{t+1} of P with Z over Y in L_{t+1}. Then for each $i=1,2,\ldots,t$, let L_i be a linear extension of P with

1. Y over Z in L_i,
2. The restriction of L_i to Z is M_i, and
3. The restriction of L_i to Y is the dual of the restriction of L_{t+1} to Y.

It is easy to see that $\mathscr{R}=\{L_1,L_2,\ldots,L_{t+1}\}$ is a realizer of P. □

Of course, the preceding proposition has a dual form in which Y is constrained to be a subset of the minimal elements of $\boldsymbol{X}$. We leave the following elementary result as an exercise.

Proposition 5.8. *Let I be a distinguishing representation of a weak order $\boldsymbol{X}=(X,P)$. Then $\dim^*(\boldsymbol{X},I)\leqslant 1$. Furthermore, $\boldsymbol{X}$ admits a distinguishing representation J with $\dim^*(\boldsymbol{X},J)=0$.*

6. Overlap graphs

Let $[a,b]$ and $[c,d]$ be closed intervals of $\mathbb{R}$. We say $[a,b]$ and $[c,d]$ *overlap* if either

1. $a<c<b<d$, or
2. $c<a<d<b$.

In other words, two intervals overlap when they intersect but neither is contained in the other.

Now let I be an interval assignment on a finite set X. Define the *overlap graph of I* as the graph $\boldsymbol{X}=(X,E)$ whose vertex set is X and whose edge set E consists of all pairs $\{x,y\}$ for which $I(x)$ and $I(y)$ overlap. An interval assignment on a set X determines both an interval order and an overlap graph. However, the overlap graph is not unique to the interval order. Instead, it depends on the representation. For the three-element interval order discussed in Section 5, the overlap graph of I is a path, while the overlap graph of J is an independent graph.

When the interval assignment I remains fixed, we use the symbol $\boldsymbol{X}$ for both the interval order and the overlap graph. Also, when $Y\subseteq X$, we use $\boldsymbol{Y}$ to denote both the induced subgraph and the subposet determined by Y. It will always be clear from the context whether we are referring to posets or graphs.

Now let I be an distinguishing interval assignment on X, and let $Y\subseteq X$. When $\boldsymbol{Y}$ is a connected subgraph of the overlap graph of I, the unique vertex of Y whose left end point is minimal in $\mathbb{R}$ is called the *root* of Y. If $y\in Y$, we let $d(x,\boldsymbol{Y})$ denote the distance in $\boldsymbol{Y}$ from x to the root of Y. In [5], Gyárfás showed that the chromatic number of an overlap graph is bounded in terms of its maximum clique size, and the following lemma is implicit in his proof.

Lemma 6.1. *Let I be a distinguishing interval assignment on a finite set X, and let $G, H \subseteq X$, with $H \subseteq G$. Suppose further that $\boldsymbol{G}$ and $\boldsymbol{H}$ are connected subgraphs of the overlap graph of I. Also, let $y \in G$, and let $[a,b]=\cup\{I(x)\colon x \in H\}$. If $d(y,\boldsymbol{G}) < d(x,\boldsymbol{G})$, for every $x \in H$, and $I(y) \bigcap [a,b] \neq \emptyset$, then $I(y) \nsubseteq [a,b]$.*

Proof. We argue by induction on $k = d(y,\boldsymbol{G})$. If $k=0$, then y is the root of G so $l_y \not\in [a,b]$. Now suppose that $k \geqslant 1$, but that $[l_y, r_y] \subseteq [a,b]$. Choose a vertex $z \in \boldsymbol{G}$ so that $d(z,\boldsymbol{G}) = k-1$ and $\{y,z\} \in E$. By the inductive hypothesis, $[l_z, r_z] \nsubseteq [a,b]$. It follows that there exists a vertex $x \in H$ which is adjacent to z in the overlap graph. This contradicts the hypothesis that all vertices in H have distance at least $k+1$ from the root of G. $\square$

7. The Proof

We are now ready to present the proof of our principal result, Theorem 1.1. In view of Proposition 4.2, Lemmas 4.4 and 5.5, it suffices to prove the following somewhat more technical result.

Theorem 7.1. *Let $m \geqslant 2$, and let I be a distinguishing representation of an interval order $\boldsymbol{X} = (X,P)$. If $\dim^*(\boldsymbol{X}, I) \geqslant 5m-8$, then $\boldsymbol{X}$ contains an m-thicket $\boldsymbol{T}_m$.*

Proof. We proceed by induction on m. First consider the base step $m=2$. Suppose that $\dim^*(\boldsymbol{X}, I) \geqslant 2$. We claim that $\boldsymbol{X}$ contains a 2-thicket. Note that a 2-thicket $\boldsymbol{T}_2$ is just $\mathbf{2}+\mathbf{1}$. If $\boldsymbol{X}$ does not contain a $\mathbf{2}+\mathbf{1}$, then it is a weak order. From Proposition 5.8, we would conclude that $\dim^*(\boldsymbol{X}, I) \leqslant 1$, which is a contradiction. Now suppose that $m \geqslant 3$ and that the theorem holds for smaller values of m.

Let $G_1, G_2, \ldots, G_s$ be the vertex sets of the components of the overlap graph determined by I. Clearly, $\dim^*(\boldsymbol{X}, I) = \max\{\dim^*(\boldsymbol{G}_i, I)\colon 1 \leqslant i \leqslant s\}$. Without loss of generality, we may assume $\dim^*(\boldsymbol{X}, I) = \dim^*(\boldsymbol{G}_1, I)$.

Set $G = G_1$, and for each $k \geqslant 0$, let $D(k) = \{x \in G\colon d(x,\boldsymbol{G}) = k\}$. Then for each $\alpha \in \{0,1\}$, let $W(\alpha) = \bigcup\{D(k)\colon k \equiv \alpha \bmod 2\}$. Then define a partition $W(\alpha) = W(\alpha, L) \cup W(\alpha, R)$ as follows. First, set $W(\alpha, L) = \{x \in W(\alpha)$: there exists a vertex $y \in G$ so that $d(y,\boldsymbol{G}) + 1 = d(x,\boldsymbol{G})$ and $l_y < l_x < r_y < r_x\}$. In other words, $I(y)$ overlaps $I(x)$ and protrudes out the *left* side of $I(x)$. Then set $W(\alpha, R) = W(\alpha) - W(\alpha, L)$. Note that if $x \in W(\alpha, R)$, and x is not the root of $\boldsymbol{G}$, then there exists a vertex $y \in G$ with $d(y,\boldsymbol{G}) + 1 = d(x,\boldsymbol{G})$ and $l_x < l_y < r_x < r_y$. Applying Lemma 5.6 twice, it follows that there exists $\alpha \in \{0,1\}$ and $\beta \in \{L,R\}$, so that $\dim^*(\boldsymbol{W}(\alpha,\beta), I) \geqslant 5m-12$. Choose a component H of $W(\alpha,\beta)$ for which $\dim^*(\boldsymbol{H}, I) = \dim^*(\boldsymbol{W}(\alpha,\beta), I)$. Note that there is some $k \geqslant 0$ so that $d(x,\boldsymbol{G}) = k$, for all $x \in H$. Furthermore, $k \geqslant 2$. For if $k \leqslant 1$, then the right-end point of the root of $\boldsymbol{G}$ is contained in the interval $I(x)$ for each $x \in H$, so $\boldsymbol{H}$ is an antichain. This implies that $\dim^*(\boldsymbol{H}, I) \leqslant 1$. In turn, this requires $5m - 12 \leqslant 1$ and thus $m \leqslant 2$. The contradiction shows $k \geqslant 2$.

Now suppose $\beta = R$. The argument is symmetric when $\beta = L$. Let Y denote the set of maximal elements of $\boldsymbol{H}$ and let $Z = H - Y$. By Lemma 5.7, $\dim^*(\boldsymbol{Z}, I) \geqslant 5m - 13$, so by the inductive hypothesis $\boldsymbol{Z}$ contains an $(m-1)$-thicket. Label the elements of this $(m-1)$-thicket as $C_{m-1} = \{x_i\colon 1 \leqslant i \leqslant m-1\}$ and $D_{m-1} = \{y_{i,j}\colon 1 \leqslant i < j \leqslant m-1\}$. Then let x_m be the unique element of Y whose left- end point is maximum. Note that $z < x_m$ in P, for all $z \in Z$.

For each $i = 1, 2, \ldots, m-1$, let $y_{i,m}$ be an element of G so that $d(y_{i,m}, \boldsymbol{G}) + 1 = d(x_i, \boldsymbol{G})$ and $l_{x_i} < l_{y_{i,m}} < r_{x_i} < r_{y_{i,m}}$. Now let $[a, b] = \bigcup\{[l_z, r_z]\colon z \in H\}$. It follows from Lemma 6.1 that $b < r_{y_{i,m}}$, for each $i = 1, 2, \ldots, m-1$. This completes the proof that $\boldsymbol{X}$ contains an m-thicket, as claimed. □

The following corollary, which follows immediately from Theorem 7.1, Proposition 4.2 and Lemma 4.4 gives a quantitative formulation of our main theorem.

Corollary 7.2. *Let* $\boldsymbol{X} = (X, P)$ *be an interval order with* $|X| = n$. *If* $\boldsymbol{Y}$ *is any interval order with* $\dim(\boldsymbol{Y}) \geqslant 30n - 6$, *then* $\boldsymbol{Y}$ *contains a subposet isomorphic to* $\boldsymbol{X}$.

8. Concluding remarks

We suspect that the linear bound in Corollary 7.2 is not tight and that the expression $30n - 6$ can be replaced by a function which is $o(n)$, perhaps as small as $O(\log n)$.

It would also be of interest to determine which interval orders have distinguishing representations for which $\dim(\boldsymbol{X}) = \dim^*(\boldsymbol{X}, I)$. We know that this is not true for weak orders, but perhaps it holds for all other interval orders.

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