A Note on Graph Pebbling

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Abstract. We say that a graph **G** is *Class* 0 if its pebbling number is exactly equal to its number of vertices. For a positive integer d, let $k(d)$ denote the least positive integer so that every graph **G** with diameter at most d and connectivity at least $k(d)$ is Class 0. The existence of the function k was conjectured by Clarke, Hochberg and Hurlbert, who showed that if the function k exists, then it must satisfy $k(d) = \Omega(2^d/d)$. In this note, we show that k exists and satisfies $k(d) = O(2^{2d})$. We also apply this result to improve the upper bound on the random graph threshold of the Class 0 property.

Key words. Pebbling, Connectivity, Threshold

1. Introduction

Let $\mathbb{N}_0$ denote the non-negative integers. When $\mathbf{G} = (V, E)$ is a finite graph, a function $\phi : V \to \mathbb{N}_0$ is called a *pebbling*. The quantity $\sum_{x \in V} \phi(x)$ is called the *size* of ϕ; the size of ϕ is just the total number of pebbles assigned to vertices. In what follows, we consider a simple rule by which one pebbling is transformed into another: Choose a vertex x to which at least two pebbles have been assigned. Remove two pebbles from x and add one pebble to an adjacent vertex y. A pebbling obtained from ϕ by a sequence of such transformations is called a *descendant* of ϕ.

Given a vertex $x \in V$ and a pebbling ϕ, we say that ϕ *pebbles* x provided $\phi(x) > 0$. Similarly, we say that ϕ has the *potential to pebble* x provided that ϕ or one of its descendants pebbles x. The *pebbling number* of a graph $\mathbf{G} = (V, E)$, denoted $f(\mathbf{G})$, is then the least p so that for any pebbling ϕ of size p and any vertex $x \in V$, ϕ has the potential to pebble x.

Clearly, the pebbling number of a graph is at least as large as the number of vertices. Following [3], we say a graph $\mathbf{G} = (V, E)$ is *Class* 0 if $f(\mathbf{G}) = |V|$. Graphs which do not belong to Class 0 are said to be *Class* 1.

In [3], Clarke, Hochberg and Hurlbert gave the following conjecture.

Conjecture 1.1. *For each $d \geq 1$, there exists a least positive integer $k(d)$ so that all graphs of diameter d and connectivity at least $k(d)$ belong to Class 0.*

This conjecture holds trivially when $d = 1$, since a graph of diameter 1 is a complete graph. Note that a path on 3 points shows that $k(2) \geq 2$. However, as noted in [3], the following example shows that $k(2) \geq 3$.

Example 1.2. Label the vertices of a 6-cycle as $x_1, \ldots, x_6$ so that $\{x_i, x_{i+1}\}$ is an edge for $i = 1, 2, \ldots, 5$. Of course, $\{x_1, x_6\}$ is also an edge. Then let $\mathbf{G}_1$ be the graph formed by adding the edges $\{x_1, x_3\}$ and $\{x_3, x_5\}$. Also, let $\mathbf{G}_2$ be the graph formed from $\mathbf{G}_1$ by adding the edge $\{x_1, x_5\}$. Then $\mathbf{G}_1$ and $\mathbf{G}_2$ are 2-connected and have diameter 2. However, the pebbling number of both graphs is 7.

Clarke, Hochberg and Hurlbert showed that $k(2) = 3$, and they characterized all 2-connected graphs with diameter 2 which belong to Class 1. The two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ constructed in the preceding example are the only such graphs on 6 or fewer vertices.

Using a "blow-up" of a path, Clarke, Hochberg and Hurlbert also showed that if the function $k(d)$ exists, then it must satisfy $k(d) = \Omega(2^d/d)$.

2. The Principal Result

In this section, we settle Conjecture 1.1 in the affirmative with the following theorem.

Theorem 2.1. *Let d be a positive integer and set $k = 2^{2d+3}$. If $\mathbf{G} = (V, E)$ is a graph of diameter at most d and connectivity at least k, then the pebbling number of $\mathbf{G}$ is $|V|$.*

Proof. Let d be a positive integer and set $k = 2^{2d+3}$. Let $\mathbf{G} = (V, E)$ be any graph with diameter at most d and connectivity at least k. Then let ϕ be any pebbling of size $|V|$ on $\mathbf{G}$. We assume that there is a vertex $z_0 \in V$ so that ϕ does not have the potential to pebble z_0 and argue to a contradiction.

We begin by defining a partition of the vertex set of $\mathbf{G}$ by setting $V = Z \cup U \cup B$, where

1. $Z = \{z \in V : \phi(z) = 0\}$,
2. $U = \{u \in V : \phi(u) = 1\}$, and
3. $B = \{b \in V : \phi(b) > 1\}$.

In the remainder of the argument, we use the natural convention that vertices of Z (*zeroes*) will be denoted by the letter z (perhaps with subscripts or primes appended). Similarly, elements of U (*units*) will be denoted by the letter u, while elements of B (*bigs*) will be denoted by the letter b. Whenever we want to make a statement about an arbitrary vertex of the graph, we will use the letter v.

Next, we observe that since the size of ϕ is $|V|$, we must have

$$\sum_{b \in B} \phi(b) = |B| + |Z|.$$

Of course the vertex z_0 belongs to Z, so $Z \neq \emptyset$. Thus $B \neq \emptyset$. Note that there are no edges from z_0 to vertices in B. Let $m = |B|$. Then let $\omega = \sum_{b \in B} \phi(b)/m$ denote the average number of pebbles assigned by ϕ to vertices in B. It follows that $|Z| = m(\omega - 1)$.

Before proceeding with the proof, we pause to make a few elementary observations about properties which the pebbling ϕ must satisfy. As noted previously, the vertex z_0 has no neighbors in B. In fact, more can be said. There cannot be a path P beginning at z_0 and ending at a point in B with all interior points of P belonging to $B \cup U$. This follows from the fact that the existence of such a path would allow us to shift a pebble from the endpoint of P which belongs to B along the path until it rests on z_0.

Also, as noted previously, we know that $\phi(v) < 2^d$ for every $v \in V$. But again we can say more.

Claim 1. *Let $v \in V$ and let $\mathscr{E}$ be a family of paths such that*

1. *Each path in $\mathscr{E}$ begins at v and ends at a point of B.*
2. *For each $b \in B$, at most $\lfloor \phi(b)/2 \rfloor$ paths in $\mathscr{E}$ end at b.*
3. *No two paths in $\mathscr{E}$ have any points in common, apart from v and, if they end at the same point, their common endpoint.*
4. *All interior points of paths in $\mathscr{E}$ belong to $B \cup U$.*

Then $|\mathscr{E}| < 2^d$.

Proof. Let $b \in B$. Then let $\mathscr{E}_b$ denote the set of all paths in $\mathscr{E}$ which end at b. We know that $|\mathscr{E}_b| \leq \lfloor \phi(b)/2 \rfloor$. It follows that we may shift $|\mathscr{E}_b|$ pebbles from b to v, one along each path in $\mathscr{E}_b$. Since the paths in $\mathscr{E}$ have no interior points in common, it follows that a descendant of ϕ places $|\mathscr{E}|$ pebbles on v. This requires $|\mathscr{E}| < 2^d$, as claimed. △

Now let $v \in V$ and consider the subgraph $\mathbf{H}_v$ of $\mathbf{G}$ induced by $\{v\} \cup U \cup B$. We modify $\mathbf{H}_v$ into a new graph $\mathbf{H}'_v$ as follows. For each vertex $b \in B - \{v\}$, we replace b by an independent set A_b of cardinality $\lfloor \phi(b)/2 \rfloor$. Furthermore, if b is adjacent to a vertex v' in $\mathbf{H}_v$, then every vertex of A_b is adjacent to v' in $\mathbf{H}'_v$, and if $b_1, b_2 \in B - \{v\}$ and b_1 is adjacent to b_2 in $\mathbf{H}_v$, then every vertex of A_{b_1} is adjacent to every vertex of A_{b_2} in $\mathbf{H}'_v$. We let $B'_v = \cup\{A_b : b \in B - \{v\}\}$. Then add a new vertex $\hat{v}$ with $\hat{v}$ adjacent to all vertices of B'_v but to no other vertices in $\mathbf{H}'_v$. In particular, $\hat{v}$ is not adjacent to v in $\mathbf{H}'_v$.

We now apply Menger's theorem to the non-adjacent pair $\{v, \hat{v}\}$, i.e., the minimum number of vertices in $\mathbf{H}'_v$ required to separate v and $\hat{v}$ is equal to the maximum number of pairwise disjoint paths from v to $\hat{v}$. Choose a minimum subset S'_v of vertices separating v from $\hat{v}$ in $\mathbf{H}'_v$. Then $\{v, \hat{v}\} \cap S'_v = \emptyset$ and every path in $\mathbf{H}'_v$ beginning at v and ending at $\hat{v}$ passes through one or more points of S'_v. Note that $S'_v \subseteq B'_v \cup U$ for every $v \in V$.

Claim 2. *For every $v \in V$, the following statements hold.*

1. *Any path in $\mathbf{H}'_v$ beginning at v and ending at a point of B'_v passes through a point of S'_v.*
2. $|S'_v| < 2^d$.
3. *If $b \in B$, $b \neq v$ and $A_b \cap S'_v \neq \emptyset$, then $A_b \subseteq S'_v$.*

Proof. Let $v \in V$. The fact that statement 1 holds is an immediate consequence of Menger's theorem and the definition of the graph $\mathbf{H}'_v$. We now show that statement 2 holds.

Let $t = |S'_v|$. Then there is a family $\mathscr{G}$ of t paths in $\mathbf{H}'_v$ each beginning at v and ending at $\hat{v}$ with any two of these paths having no point in common other than v and $\hat{v}$. For each path P in $\mathscr{G}$, start at v and travel along P towards $\hat{v}$. Then let $b(P)$ be the first point on P which belongs to B'_v. Clearly, there must be such a point since the next to last point of P belongs to B'_v. Note that any point of P between v and $b(P)$ on P belongs to U. Then let P' denote the initial segment of P beginning at v and ending at $b(P)$, and let $\mathscr{G}' = \{P' : P \in \mathscr{G}\}$. Note that by Menger's theorem, each path in $\mathscr{G}'$ contains a unique point of S'_v.

For each $b \in B - \{v\}$, let $\mathscr{G}'_b$ denote the set of all paths in $\mathscr{G}'$ which end at a point of A_b, and let B_v consist of those points of $B - \{v\}$ for which $\mathscr{G}'_b \neq \emptyset$. Clearly, $|\mathscr{G}'_b| \leq |A_b| = \lfloor \phi(b)/2 \rfloor$ for each $b \in B_v$.

For each $b \in B_v$, let $\mathscr{G}_b$ denote the family of paths in $\mathbf{H}_v$ obtained by replacing the ending point of each path in $\mathscr{G}'_b$ by b. Evidently, all paths in $\mathscr{G}_b$ start at v and end at b. However, other than starting and ending points, the paths in $\mathscr{G}_b$ are pairwise disjoint. Also, if b_1 and b_2 are distinct elements of B_v, then v is the unique common point of a path from $\mathscr{G}_{b_1}$ and a path from $\mathscr{G}_{b_2}$. From these remarks, it follows from Claim 1 that $|S'_v| = |\mathscr{G}| = |\mathscr{G}'| < 2^d$, as claimed.

Finally, we show that statement 3 holds. Let $b \in B - \{v\}$ and suppose that $A_b \cap S'_v \neq \emptyset$. Choose a path P in $\mathscr{G}$ having a point $b_1 \in A_b \cap S'_v$ in its interior. Then b_1 is the unique point from S'_v belonging to P. If there is a point $b_2 \in A_b - S'_v$, then the path $\hat{P}$ obtained from P by appending b_2 and $\hat{v}$ onto the initial segment of P beginning at v and ending at the vertex immediately preceding b_1 on P is a path from v to $\hat{v}$ in $\mathbf{H}'_v$ which passes through no point of S'_v. This would contradict the assumption that S'_v separates v from $\hat{v}$. So $A_b \subseteq S'_v$ as claimed. $\triangle$

For each $v \in V$, let $S_v = (S'_v \cap U) \cup \{b \in B - \{v\} : A_b \subseteq S'_v\}$. Note that $v \notin S_v$. Also note that S_v separates v from $B - \{v\}$ in $\mathbf{H}_v$, i.e., any path in $\mathbf{H}_v$ starting at v and ending at a point in $B - \{v\}$ passes through one or more points of S_v. From Claim 2, it is clear that $|S_v| < 2^d$. However, using the second part of Claim 2, even more can be said.

Claim 3. *For every $v \in V$,*

$$\phi(v) + \sum_{b \in S_v} \phi(b) < 2^{d+2}.$$

Proof. Let $v \in V$. Obviously, $\phi(v) < 2^d$. From Claim 2, we know that

$$\sum_{b \in S_v} \lfloor \phi(b)/2 \rfloor = \sum_{b \in S_v} |A_b| \leq |S_v'| < 2^d$$

and

$$\sum_{b \in S_v} \phi(b) \leq 2 \sum_{b \in S_v} \left(\lfloor \phi(b)/2 \rfloor + |S_v'| \right) \leq 3(2^d),$$

so that

$$\phi(v) + \sum_{b \in S_v} \phi(b) < 2^d + 3(2^d) = 2^{d+2}. \qquad \triangle$$

Claim 4. *There exists a positive integer q with $q > m\omega/2^{d+2}$, a q-element subset $B_0 \subset B$, and a labelling $\{b_1, b_2, \ldots, b_q\}$ of the elements of B_0 so that for all i and j with $1 \leq i < j \leq q$, $b_j \notin S_{b_i}$.*

Proof. We form the subset B_0 inductively. Choose an arbitrary element of B as b_1. Then remove from consideration all remaining elements $S_{b_1} \cap B$. By Claim 3, the total number of pebbles assigned by ϕ to the elements selected or removed from consideration is less than 2^{d+2}. Since the total number of pebbles assigned to elements of B is $m\omega$, we may repeat this procedure $m\omega/2^{d+2}$ times to obtain the desired subset B_0. $\qquad \triangle$

The reader should note that if $1 \leq i < j \leq q$, then we know that $b_j \notin S_{b_i}$, but we do not know whether b_i belongs to S_{b_j}. Now let $W = \cup_{b \in B_0} S_b$. Note that $|W| < q2^d$. Also note that $W \cap Z = \emptyset$.

Since the connectivity of **G** is at least k, we know that for each $b \in B_0$, there are k paths $P_1(b, z_0), P_2(b, z_0), \ldots, P_k(b, z_0)$, each beginning at b and ending at z_0, with two paths in this family having no points in common other than b and z_0. Since $\{b, z_0\}$ is not an edge in **G**, each path $P_i(b, z_0)$ contains at least one interior point, and in fact, each $P_i(b, z_0)$ contains at least one interior point which belongs to Z.

Since $|S_b| < 2^d$ and $k = 2^{2d+3}$, we may assume that the paths have been labelled so that for each $i = 1, 2, \ldots, k - 2^d$, the path $P_i(b, z_0)$ does not contain a point of S_b.

Now let $b \in B_0$ and let i be an integer with $1 \leq i \leq k - 2^d$. Follow the path $P_i(b, z_0)$ beginning at b and let $v_i(b)$ be the first point (distinct from b) on the path which belongs to $W \cup Z$. We let $Q_i(b)$ be the initial segment of $P_i(b, z_0)$ beginning at b and ending at $v_i(b)$. We call b the *root* of this path and $v_i(b)$ the *terminal point*, and we let $\mathscr{F} = \{Q_i(b) : b \in B_0, 1 \leq i \leq k - 2^d\}$. Note that $|\mathscr{F}| = q(k - 2^d)$.

Of course, for each $v \in W \cup Z$ and each $b \in B$, there is at most one integer i with $1 \leq i \leq k - 2^d$ for which the path $Q_i(b)$ has b as its root and v as its terminal point. However, if $b, b' \in B$, there may exist integers $i, i' \in \{1, 2, \ldots, k - 2^d\}$ so that $Q_i(b)$ and $Q_{i'}(b')$ both have v as their terminal point; but when b and b' both belong to B_0, we can say much more.

Claim 5. *Let* $v \in W \cup Z$ *and let* b *and* b' *be distinct points of* B_0. *If* $i, i' \in \{1, 2, \ldots, k - 2^d\}$ *and* $Q_i(b)$ *and* $Q_{i'}(b')$ *both have* v *as their terminal point, then* $Q_i(b)$ *and* $Q_{i'}(b')$ *have no point in common other than* v.

Proof. Suppose that $F = Q_i(b)$ and $F' = Q_{i'}(b')$ have a common point distinct from v. Then there exists a path F'' from b to b' with $F'' \subseteq (F \cup F') - \{v\}$. Furthermore, $F'' \subseteq B \cup U$. Thus $F'' \cap S_b \neq \emptyset \neq F'' \cap S_{b'}$. However, $F'' \cap F$ does not contain a point from S_b, and $F'' \cap F'$ does not contain a point from $S_{b'}$. Therefore $F'' \cap F \cap S_{b'} \neq \emptyset$ and $F'' \cap F' \cap S_b \neq \emptyset$.

Without loss of generality, we may assume that b was chosen before b' in the construction of B_0 described in Claim 3. Then $b' \notin S_b$. It follows that $F'' \cap F' \cap S_b$ contains a point of S_b which is distinct from b'. However, this contradicts the hypothesis that v is the terminal point of F'.   △

We pause to point out that the argument in the preceding claim works only in one direction, as it may happen that $b \in S_{b'}$.

We are now ready to complete the proof. We observe that $|W \cup Z| < q2^d + m(\omega - 1) < q(2^{d+2} + 2^d)$ and $|\mathscr{F}| = q(k - 2^d)$. Since $|\mathscr{F}|/|W \cup Z| > 2^d - 1$, it follows that there is a vertex $v \in W \cup Z$ and a subfamily $\mathscr{E} \subset \mathscr{F}$, with $|\mathscr{E}| = 2^d$, so that every path in $\mathscr{E}$ has v as its terminal point. However, from Claim 5, any two paths from $\mathscr{E}$ have no point in common other than v, so the existence of $\mathscr{E}$ is now seen to be a contradiction to Claim 1.   □

3. Threshold

The notion that graphs with very few edges tend to have large pebbling number and graphs with very many edges tend to have small pebbling number can be made precise as follows. Let $\mathscr{G}_{n,p}$ be the random graph model in which each of the $\binom{n}{2}$ possible edges of a random graph having n vertices appears independently with probability p. For functions f and g on the natural numbers we write that $f \ll g$ (or $g \gg f$) when $f/g \to 0$ as $n \to \infty$. Let $o(g) = \{f \mid f \ll g\}$ and define $O(g)$ (resp., $\Omega(g)$) to be the set of functions f for which there are constants c, N such that $f(n) \leq cg(n)$ (resp., $f(n) \geq cg(n)$) whenever $n > N$. Finally, let $\Theta(g) = O(g) \cap \Omega(g)$.

Let $\mathscr{P}$ be a property of graphs and consider the probability $Pr(\mathscr{P})$ that the random graph $\mathscr{G}_{n,p}$ has $\mathscr{P}$. For large p it may be that $Pr(\mathscr{P}) \to 1$ as $n \to \infty$, and for small p it may be that $Pr(\mathscr{P}) \to 0$ as $n \to \infty$. More precisely, define the *threshold of* $\mathscr{P}$, $th(\mathscr{P})$, to be the set of functions t for which $p \gg t$ implies that $Pr(\mathscr{P}) \to 1$ as $n \to \infty$, and $p \ll t$ implies that $Pr(\mathscr{P}) \to 0$ as $n \to \infty$, if this set is nonempty.

It is not clear that such thresholds exist for arbitrary $\mathscr{P}$. However, we observe that Class 0 is a monotone property (adding edges to a Class 0 graph maintains the property), and a theorem of Bollobás and Thomason [1] states that $th(\mathscr{P})$ exists for every monotone $\mathscr{P}$. It is well known [4] that $th(\text{connected}) = \Theta(\lg n/n)$, and since connectedness is required for Class 0, we see that $th(\text{Class } 0) \subseteq \Omega(\lg n/n)$. In [3] it is

noted that $\mathcal{G}_{n,1/2}$ is Class 0 with probability tending to 1. It is straightforward to extend their argument to show that, for fixed p, $\mathcal{G}_{n,p}$ is Class 0 with probability tending to 1. Here we prove the following theorem.

Theorem 3.1. *For all $d > 0$, $th(\text{Class }0) \subseteq o((n \lg n)^{1/d}/n)$.*

Proof. We prove the equivalent statement that $th(\text{Class }0) \subseteq O((n \lg n)^{1/d}/n)$ for all $d > 0$. It is proven in [2] that $th(\text{diameter} \leq d) \subseteq \Omega((n \lg n)^{1/d}/n)$, and in [5] that $th(\text{connectivity} \geq k) \subseteq \Omega((\lg n + k \lg \lg n)/n)$. Hence, for any fixed d and k with $k \geq 2^{2d+3}$, and for any $p \gg (n \lg n)^{1/d}/n$, the probability that $\mathcal{G}_{n,p}$ has diameter at most d and connectivity at least k tends to 1. Therefore the probability that $\mathcal{G}_{n,p}$ is Class 0 tends to 1. $\square$

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