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SOME THEOREMS ON GRAPHS AND POSETS

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In this journal, Leclerc proved that the dimension of the partially ordered set consisting of all subtrees of a tree T, ordered by inclusion, is the number of end points of T. Leclerc posed the problem of determining the dimension of the partially ordered set P consisting of all induced connected subgraphs of a connected graph G for which P is a lattice.

In this paper, we prove that the poset P consisting of all induced connected subgraphs of a nontrivial connected graph G, partially ordered by inclusion, has dimension n where n is the number of noncut vertices in G whether or not P is a lattice. We also determine the dimension of the distributive lattice of all subgraphs of a graph.

1. Introduction

The dimension of a partially ordered set (X, P) was defined by Dushnik and Miller [4] as the minimum number of linear orders on X whose intersection is P. A poset has dimension one if and only if it is a chain. In [6] Leclerc proved the following result which gives an alternate definition of dimension for those posets which are not chains.

Theorem 1.1. *The dimension of a poset (X, P) which is not a chain is the smallest positive integer n for which there exists a tree T with n end vertices so that (X, P) is isomorphic to some collection of subtrees of T ordered by inclusion.*

In this paper we use the characterization of dimension in the authors' paper [13] to generalize Theorem 1.1 to arbitrary graphs.

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2. TM-cycles

In this section, we briefly summarize some results in [13]. For a poset (X, P), we denote the set of all incomparable pairs $\{(x, y): xIy \text{ in } P\}$ by $\mathcal{I}_P$. For a subset $Q \subseteq X \times X$, we denote the transitive closure of Q by $\overline{Q}$.

Theorem 2.1. *Let $\mathcal{S} \subseteq \mathcal{I}_P$. Then $\overline{P \cup \mathcal{S}}$ is a partial order on X iff there does not exist a subset $\{(a_i, b_i): 1 \leqslant i \leqslant m\} \subseteq \mathcal{S}$ where $\{(b_i, a_{i+1}): 1 \leqslant i \leqslant m\} \subseteq P$.*

We note that in the statement of Theorem 2.1 it is necessary to interpret the subscripts cyclically, i.e., $a_{m+1} = a_1$. If $\mathcal{S} \subseteq \mathcal{I}_P$, a subset of the form $\{(a_i, b_i): 1 \leqslant i \leqslant m\} \subseteq \mathcal{S}$ for which $(b_i, a_j) \in P$ iff $j = i + 1$ for $i = 1, 2, ..., m$, is called a TM-cycle of length m (for $\mathcal{S}$).

Corollary 2.2. *Let $\mathcal{S} \subseteq \mathcal{I}_P$. Then $\overline{P \cup \mathcal{S}}$ is a partial order iff $\mathcal{S}$ has no TM-cycles.*

This corollary and Szpilrajn's well known theorem [7] on the extension of partial orders to linear orders gives the following alternate definition of dimension for posets for which $\mathcal{I}_P \neq \emptyset$. (Note that if $\mathcal{I}_P = \emptyset$, then (X, P) is a chain and $\dim(X, P) = 1$.)

Theorem 2.3. *The dimension of a poset (X, P) for which $\mathcal{I}_P \neq \emptyset$ is the smallest positive integer t for which there exists a partition $\mathcal{I}_P = \mathcal{S}_1 \cup \mathcal{S}_2 \cup ... \cup \mathcal{S}_t$ so that no $\mathcal{S}_i$ has a TM-cycle.*

3. Crowns and irreducible posets

A poset is said to be irreducible if $\dim Y < \dim X$ for every proper subposet. An n-dimensional irreducible poset is said to be n-irreducible; by convention, we consider a one point poset to be 1-irreducible. The only 2-irreducible poset is a two element antichain. However, the problem of finding the collection of all n-irreducible posets has not been solved for any $n \geqslant 3$. A list of all known 3-irreducible posets and many examples of n-irreducible posets for $n \geqslant 4$ are given in [11].

An n-dimensional poset contains an n-irreducible subposet. In this paper we will prove that a poset X has dimension at least n where $n \geqslant 3$

by exhibiting an n-irreducible subposet of X. Usually this n-irreducible poset will be the "standard" example of an n-irreducible poset – the crown S_n^0. Using the terminology and notation introduced in [8], the crown S_n^0 is defined (for $n \geqslant 3$) as the poset of height one with n maximal elements $a_1, a_2, ..., a_n$ and n minimal elements $b_1, b_2, ..., b_n$; the partial order on S_n^0 is defined by $b_i < a_j$ if and only if $i \neq j$. S_n^0 is isomorphic to the set of one-element and $(n-1)$-element subsets of an n-element set ordered by inclusion.

For the sake of completeness, we include here an argument to show that S_n^0 is indeed an n-dimensional poset. We refer the reader to [3] and [8] for much more general results for families of posets which include S_n^0. Characterization theorems involving S_n^0 are given in [1,5,10,12].

Fact. $\dim S_n^0 = n$.

Proof. Let P be the partial order on S_n^0 and suppose that $\mathcal{G}_P = \mathcal{S}_1 \cup \mathcal{S}_2 \cup ... \cup \mathcal{S}_t$, where $t < n$. It follows that there exist distinct integers i and j so that $(a_i, b_i) \in \mathcal{S}_k$ and $(a_j, b_j) \in \mathcal{S}_k$ for some $k \leqslant t$. But this implies that $\mathcal{S}_k$ contains a TM-cycle of length 2. We conclude that $\dim S_n^0 \geqslant n$.

On the other hand for each $i \leqslant n$, let $\mathcal{S}_i = \{(a_i, b_i)\}$. Since $|\mathcal{S}_i| = 1$, $\mathcal{S}_i$ cannot have any TM-cycles and then $Q_i = P \cup \mathcal{S}_i$ is a partial order for each $i \leqslant n$. Now let $L_1, L_2, ..., L_n$ be arbitrary linear orders on S_n^0 so that $Q_i \subset L_i$. It is easy to see that P is the intersection of $L_1, L_2, ..., L_n$ and thus $\dim S_n^0 \leqslant n$.

4. Induced subgraphs of connected graphs

The induced connected subgraphs of a connected graph G ordered by inclusion form a poset which we denote $X(G)$. In [6] Leclerc posed the question of determining the dimension of $X(G)$ for those graphs for which $X(G)$ is a lattice. In this section, we determine the dimension of $X(G)$ for an arbitrary connected graph G (including those for which $X(G)$ is not a lattice). Our theorem will use the standard graph theoretic concepts of the distance between vertices and the distance between two sets of vertices. The distance between a vertex x and itself, $d(x, x)$, is zero while if x and y are distinct vertices in a connected graph, the distance from x to y, $d(x, y)$, is one less than the minimum number of vertices in a path from x to y (including x and y). If x is a vertex and A is a set of vertices, the distance from x to A, $d(x, A)$, is $\min\{d(x, a): a \in A\}$.

Theorem 4.1. *If G is a nontrivial connected graph with n non-cut vertices, then* $\dim X(G) = n$.

Proof. Let P denote the partial order on $X(G)$. Then label the non-cut vertices of G, $x_1, x_2, \ldots, x_n$. Since G is non-trivial, we note that $n \geq 2$. We first prove that $\dim X(G) \geq n$. If $n = 2$, it is clear that $\dim X(G) \geq 2$, and if $n \geq 3$, then the collection of connected subgraphs of G consisting of the trivial graphs determined by the non-cut vertices and the induced subgraphs of the form $G - x_i$, $i = 1, 2, \ldots, n$, form a copy of S_n^0. We conclude that $\dim X(G) \geq n$.

To show that $\dim X(G) \leq n$, we partition the set of incomparable pairs $\mathcal{I}_P$ into n subsets so that no subset has a TM-cycle. We define for each $i \leq n$, $\mathcal{S}_i = \{(H_1, H_2) \in \mathcal{I}_P : d(x_i, H_2) < d(x_i, H_1)\}$.

We first show that $\mathcal{I}_P = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots \cup \mathcal{S}_n$. Suppose $(H_1, H_2) \in \mathcal{I}_P$. Then there exists a vertex $x \in H_2 - H_1$. If x is a non-cut vertex of G, say $x = x_i$, then $(H_1, H_2) \in \mathcal{S}_i$. Now suppose x is a cut vertex of G. Then H_1 is a subgraph of one of the components of $G - x$. Choose a non-cut vertex v_i of G which is in a component of $G - x$ not containing H_1. It follows that $(H_1, H_2) \in \mathcal{S}_i$.

We now show that none of these subsets of $\mathcal{I}_P$ contain a TM-cycle. Suppose that $\{(H_j, K_j) : 1 \leq j \leq m\}$ is a TM-cycle for $\mathcal{S}_i$. Then it follows that

$$\begin{aligned} d(x_i, K_1) &< d(x_i, H_1) \leq d(x_i, K_m) < d(x_i, H_m) \leq d(x_i, K_{m-1}) \\ &< d(x_i, H_{m-1}) < \ldots < d(x_i, H_2) \leq d(x_i, K_1) . \end{aligned}$$

The contradiction completes the proof.

5 Distributive lattices and subgraphs

Dilworth [2] proved that the dimension of a distributive lattice L is the width of the subposet of L consisting of the join irreducibles. If L is n-dimensional (for $n \geq 3$), then L contains S_n^0 as a subposet. We refer the reader to [9] for a simple proof of Dilworth's theorem.

If G is a graph, then the poset consisting of all subgraphs (not just the induced subgraphs) of G ordered by inclusion is a distributive lattice. The join irreducible elements of this lattice are those subgraphs consisting of a single vertex or two vertices joined by an edge.

Theorem 5.1. *Let G be a graph with components, $C_1, C_2, \ldots, C_t$. For each $i \leq t$, let p_i and q_i be the number of vertices and edges respectively of C_i. Then the dimension of the distributive lattice of all subgraphs of G ordered by inclusion is*

$$\sum_{i=1}^{t} \max\{p_i, q_i\} .$$

Proof. We denote the lattice of all subgraphs of G by $S(G)$ and the subposet of $S(G)$ consisting of all join irreducible elements by $P(G)$. If G has t components, $C_1, C_2, \ldots, C_t$ where $t \geq 2$. then $P(G)$ is the free sum $P(C_1) + P(C_2) + \ldots + P(C_t)$. Therefore a maximum antichain A in $P(G)$ is the free sum $A = A_1 + A_2 + \ldots + A_t$ where each A_i is a maximum antichain in $P(C_i)$. Hence it suffices to prove our theorem for connected graphs.

Let G be a nontrivial connected graph with p vertices and q edges. Choose a maximum antichain A from $P(G)$ so that the number of edges in A is as large as possible. If A contains all the edges of G, then $|A| = q$; on the other hand, if A contains no edges of G, then $|A| = p$. Now suppose that A contains some but not all of the edges in G. Since G is connected we may choose coincident edges e and f with $e \in A$ but $f \notin A$. Let the end vertices of e be x and y and the end vertices of f be y and z. It follows that $y \notin A$ and $z \in A$; we may then conclude that $A - \{z\} \cup \{f\}$ is a maximum antichain of $P(G)$ containing one more edge than A. The contradiction completes the proof.

Note added in proof. The collection of all 3-irreducible posets has been independently determined by the authors and D. Kelly.

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