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ORIGINAL PAPER

Hamiltonian Cycles and Symmetric Chains in Boolean Lattices

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Abstract Let $B(n)$ be the subset lattice of $\{1, 2, \ldots, n\}$. Sperner's theorem states that the width of $B(n)$ is equal to the size of its biggest level. There have been several elegant proofs of this result, including an approach that shows that $B(n)$ has a symmetric chain partition. Another famous result concerning $B(n)$ is that its cover graph is hamiltonian. Motivated by these ideas and by the Middle Two Levels conjecture, we consider posets that have the Hamiltonian Cycle–Symmetric Chain Partition (HC-SCP) property. A poset of width w has this property if its cover graph has a hamiltonian cycle which parses into w symmetric chains. We show that the subset lattices have the HC-SCP property, and we obtain this result as a special case of a more general treatment.

Keywords Hamiltonian cycle · Symmetric chain decomposition · Boolean lattice · Cartesian product · Cover graph

Mathematics Subject Classification 06A07 · 06E10

1 Introduction

For a positive integer n, we let $\mathcal{B}(n)$ denote the subset lattice consisting of all subsets of $[n]$ ordered by inclusion. Of course, we may also consider $\mathcal{B}(n)$ as the set of all 0–1 strings of length n with partial order

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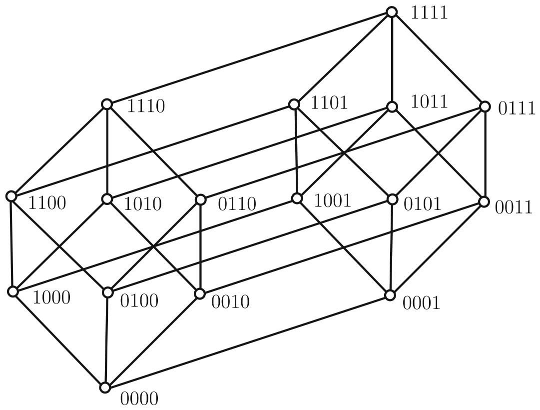
Fig. 1 The subset lattice $\mathcal{B}(4)$



$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \leq \mathbf{b} = (b_1, b_2, \ldots, b_n)$$

if and only if $a_i \leq b_i$ for each $i = 1, 2, \ldots, n$. We illustrate this with a diagram for $\mathcal{B}(4)$ in Fig. 1.

Some elementary properties of the poset $\mathcal{B}(n)$ are:

1. The height is $n + 1$ and all maximal chains have exactly $n + 1$ points.
2. The size of the poset $\mathcal{B}(n)$ is 2^n.
3. The elements are partitioned into ranks (antichains) $A_0, A_1, \ldots, A_n$, where A_i is the $\binom{n}{i}$ subsets of $[n]$ of size i, for each $i = 0, 1, \ldots, n$.
4. The maximum size of a rank in $\mathcal{B}(n)$ occurs in the middle, i.e. at $\binom{n}{\lfloor n/2 \rfloor}$. Note that when n is odd, there are two ranks of maximum size, but when n is even, there is only one.

For the width of the subset lattice, we have the following classic result due to Sperner [15].

Theorem 1 (Sperner) *For each* $n \geq 1$, *the width of the subset lattice* $\mathcal{B}(n)$ *is the maximum size of a rank, i.e.,*

$$\text{width}(\mathcal{B}(n)) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

With the aim of extending Sperner's theorem to the lattice of divisors, de Bruijn et al. [2] introduced symmetric chain decompositions. Later, Katona [8] and Kleitman [9, 10] used symmetric chains to reprove Sperner's theorem and to attack the Littlewood-Offord problem [11]. Subsequently, the study of symmetric chain partitions of the Boolean lattice has played a large role in combinatorics (e.g. [5,6]). For much more on symmetric chains and how they relate to the concepts here, see [1,3].

Next, we present a concise summary of proof of Sperner's theorem using symmetric chains in order to motivate and prepare for our results.

1.1 Proving Sperner with Symmetric Chains

A poset **P** is said to be *ranked* if all maximal chains have the same cardinality. When a poset is ranked, then there is a partition $X = A_1 \cup A_2 \cup \ldots A_h$ so that every maximal chain consists of exactly one point from each A_i. We call this partition its *partition into ranks*, and we call each A_i a *rank*.

A ranked poset is said to be *Sperner* if the width of the poset is just the maximum cardinality of a rank. So using this terminology, Sperner's theorem is just the assertion that the subset lattice is Sperner.

Let **P** be a ranked poset of height h and let $A_1, A_2, \ldots, A_h$ be the ranks of **P**. A chain C in **P** is called a *symmetric chain* if there exists an integer s so that C contains exactly one point from each rank $A_s, A_{s+1}, \ldots, A_{h+1-s}$. Therefore, (1) C is balanced about the middle of the poset and (2) it is not possible to insert a point in between two consecutive points in C.

We say a poset has a *SCP* when its elements can be partitioned into symmetric chains. (In the literature, this is often *SCD* for "symmetric chain decomposition.") The following proposition is self-evident.

Proposition 2 *If a ranked poset has a SCP, then it is Sperner and its width is the size of its middle rank(s).*

So an alternative proof of Sperner's theorem is provided by the following result, due independently to de Bruijn et al. [2], Katona [8], and Kleitman [9].

Theorem 3 *For each* $n \geq 1$, *the subset lattice* $\mathcal{B}(n)$ *has a SCP.*

In fact, a stronger result can be established. But first, we need a definition. Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ be posets. The *cartesian product* $\mathbf{P} \times \mathbf{Q}$ is the poset with ground set $X_P \times X_Q$ and partial order $\{((a_1, b_1), (a_2, b_2)) \mid (a_1, a_2) \in P$ and $(b_1, b_2) \in Q\}$. We can now state the stronger result.

Theorem 4 *If* **P** *and* **Q** *are ranked posets and each has a SCP, then* $\mathbf{P} \times \mathbf{Q}$ *is ranked and has a SCP.*

Note that Theorem 3 follows immediately from Theorem 4 since $\mathcal{B}(n)$ is just the cartesian product of n copies of the two-element chain **2**, and **2** has a trivial SCP. Readers who are familiar with the elementary proofs that the family of posets which have SCP's is closed under cartesian products will recognize that similar concepts are being applied here, although there are additional hurdles that have to be surmounted.

The remainder of the paper is organized as follows. In Sect. 2, we define a class of posets that generalizes ranked posets; namely, leveled posets. In Sect. 3, we define the HC-SCP property and the strong HC-SCP property, the primary definitions in this work. In Sect. 4, we prove:

Theorem 5 *For each* $n \geq 1$, *the subset lattice* $\mathcal{B}(n)$ *satisfies the strong HC-SCP property.*

Sections 5, 6, and 7 are dedicated to the proof of the following theorem.

Theorem 6 *Let* **P** *and* **Q** *be posets with the strong HC-SCP property. Then* $\mathbf{P} \times \mathbf{Q}$ *has the HC-SCP property.*

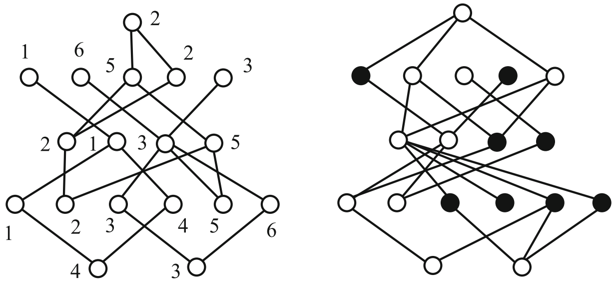


Fig. 2 Two leveled posets—only one is Sperner

2 Leveled Posets

A slightly more general class of posets than ranked posets is leveled posets. A poset $\mathbf{P}$ is *leveled* if there is a partition $\mathbf{P} = A_1 \cup A_2 \cup \cdots \cup A_h$, where h is the height of $\mathbf{P}$, with each A_i an antichain, so that if x covers y in $\mathbf{P}$, there is some $i \geq 2$ for which $x \in A_i$ and $y \in A_{i-1}$. Naturally, we refer to the antichains as *levels*, and note that the width of $\mathbf{P}$ is at least as large as the maximum size of a level. Ranked posets, and in particular the subset lattices, are leveled. However, not all leveled posets are ranked. When a leveled poset $\mathbf{P}$ is a connected (i.e. its cover graph is connected) the antichain partition is unique. In the treatment to follow, we will only consider connected posets.

A leveled poset is called *Sperner* when its width is the maximum size of a level. In Fig. 2, we show two leveled posets. In both, the sizes of the levels are 1, 2, 4, 5 and 6. The poset on the left is Sperner, and the numbers on the figure indicate a partition into six chains. The poset on the right is not Sperner, as the darkened points form an antichain of size 8.

Let $\mathbf{P}$ be a leveled poset of height h. A chain $C = \{x_1 < x_2 < \cdots < x_m\}$ is *symmetric* if (1) $x_1 \in A_i$ implies that $x_m \in A_{h+1-i}$, and (2) x_{i+1} covers x_i for each $i = 1, 2, \ldots, m-1$. As before, a SCP of $\mathbf{P}$ is a partition into symmetric chains. The following proposition is self-evident.

Proposition 7 *Let* $\mathbf{P}$ *be a leveled poset. If* $\mathbf{P}$ *has a SCP, then* $\mathbf{P}$ *is Sperner. Furthermore, if* h *is the height of* $\mathbf{P}$, *then* $|R_i| = |R_{h+1-i}|$, *for every* $i = 1, 2, \ldots, \lfloor h/2 \rfloor$. *Moreover, if* $h = 2r + 1$, *then the width of* $\mathbf{P}$ *is* $|R_{r+1}|$, *and if* $h = 2r$, *then the width of* $\mathbf{P}$ *is* $|R_r| = |R_{r+1}|$.

In Fig. 3, we show a leveled poset, with some of the covers shown as dashed lines (we will explain this detail shortly), and the points have been arranged so that the vertical chains form a symmetric chain partition.

3 Symmetric Chain Partitions and Hamiltonian Cycles

The cover graph $Q(n)$ of the subset lattice $\mathcal{B}(n)$ is called a *cube*. These graphs have been studied extensively, as they exhibit many interesting combinatorial properties.

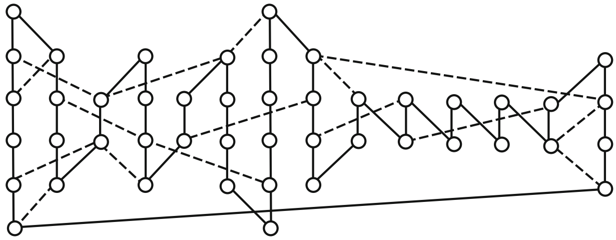


Fig. 3 Symmetric chain partition of a leveled poset

One such property is that of being *hamiltonian*, meaning that there is a cycle in the graph that includes all of the vertices. The following is a combinatorial gem, a result frequently used in elementary combinatorics classes to illustrate the power of induction.

Theorem 8 *For $n \geq 2$, the cube $Q(n)$ is hamiltonian.*

Hamiltonian cycles in Boolean lattices are commonly referred to as Gray codes. Finding Gray codes in subgraphs of $Q(n)$ has been the subject of extensive study due to their wide array of applications, including engineering, cryptography, genetic algorithms, and error correction. For a survey of Gray codes and their uses, see [12,17].

3.1 Hamiltonian Cycle–Symmetric Chain Partition Property

Take a second look at Fig. 3 and observe that the covers which are shown by solid lines form a hamiltonian cycle in the cover graph of **P**. We say that a leveled poset has the Hamiltonian Cycle–Symmetric Chain Partition property, which we abbreviate as the HC-SCP property, if its cover graph has a hamiltonian cycle which parses into a SCP. Eventually, we will show that the subset lattice has the HC-SCP property, but we elect to obtain this result as a special case of a more general treatment, just as de Bruijn et al., Katona and Kleitman did for the symmetric chain property.

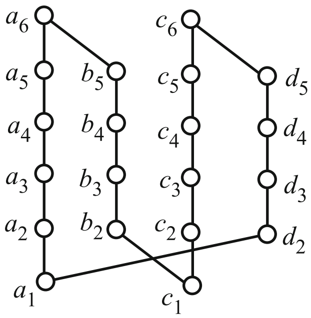
3.2 The Special Role of a 2-Element Chain

By convention, we say that a connected graph on two vertices has a hamiltonian cycle in the sense that all vertices can be listed so that (1) no vertex appears twice in the list, (2) consecutive vertices are adjacent, and (3) the last vertex is also adjacent to the first. So with this convention, we could say that the cube $Q(n)$ is hamiltonian for all $n \geq 1$.

We want to develop a framework for studying leveled posets with the HC-SCP property so that if **P** and **Q** have this property, so does $\mathbf{P} \times \mathbf{Q}$. We begin modestly, with $\mathbf{Q} \cong \mathbf{2}$. Still, there are challenges to achieving this goal.

Example 9 Consider the leveled poset **P** shown in Fig. 4. Evidently **P** has the HC-SCP property. However, it is a simple exercise to show that the cartesian product $\mathbf{P} \times \mathbf{2}$ does not. We leave this to the reader.

Fig. 4 A troublesome poset



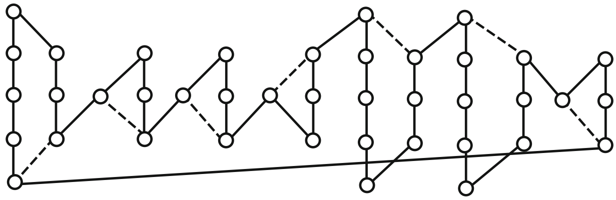


Fig. 5 A leveled poset of odd height satisfying the HC-SCP property

3.3 A Stronger Property

Let **P** be a leveled poset and let h and w denote respectively, the height and width of **P**. We say that **P** satisfies the *strong* HC-SCP property when there is a hamiltonian cycle H in the cover graph of **P** so that (1) H parses into a symmetric chain partition consisting of the chains $C_1, C_2, \dots, C_w$ labeled in the order they are encountered in traveling around H and (2) the chains in this partition can be partitioned into non-empty blocks $B_1, B_2, \dots, B_s$, with all chains in a block occurring consecutively (in the cyclic sense) in H, so that for each $i = 1, 2, \dots, s$, one of the following statements applies:

1. $|B_i| = 2$, and if $B_i = \{C, C'\}$ with $|C'| = 2 + |C|$, then the least element of C covers the least element of C' and the greatest element of C is covered by the greatest element of C'.
2. h is even and B_i consists of a single 2-element chain.

We refer to blocks as Type 1 or Type 2 according to whether the first or the second of these two statements holds. When h is odd, note that any poset satisfying the HC-SCP property has even width, and only Type 1 blocks can be used. We show in Fig. 5 a leveled poset of height five satisfying the strong HC-SCP property. When h is even, a poset satisfying the HC-SCP property can have even width or odd width. In Fig. 6, we show a leveled poset of height six which satisfies the strong HC-SCP property. Here there are four Type 2 blocks. Note that not all 2-element chains form Type 2 blocks. Some of them may be absorbed in Type 1 blocks with the other chain having four elements.

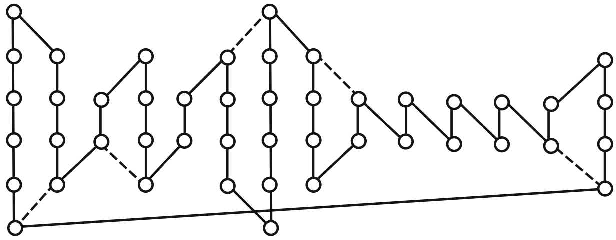


Fig. 6 A leveled poset of even height satisfying the HC-SCP property

4 The Strong Theorem

We are now positioned to prove the following structural theorem.

Theorem 10 *Let* $\mathbf{P}$ *be a leveled poset. If* $\mathbf{P}$ *satisfies the strong HC-SCP property, so does the cartesian product* $\mathbf{P} \times \mathbf{2}$.

Proof We start with a hamiltonian cycle H in $\mathbf{P}$ that parses into symmetric chains, with $B_1, B_2, \ldots, B_s$ the partition witnessing that $\mathbf{P}$ satisfies the strong HC-SCP property. We will then proceed to construct the required hamiltonian cycle H' in $\mathbf{P} \times \mathbf{2}$, together with the specification of the blocks which show that $\mathbf{P} \times \mathbf{2}$ also satisfies the strong HC-SCP property. We find it useful to use the following natural notation and terminology. A set of points which occur consecutively in a hamiltonian cycle will be called a *group*. Abusing notation slightly, we will also consider each block B_i as a group in H, so we will talk about H entering the block B_i at a point $x \in C$ from B_i and leaving it at a point $y \in C'$ from B_i. Note that when B_i is a Type 2 block, the points x and y are just the top and bottom points of the same chain.

Our construction for H' will satisfy the following properties.

1. For each $i = 1, 2, \ldots, s$, the elements of $G_i = \bigcup\{C \times \{0, 1\} : C \in B_i\}$ will be a group in H'.
2. If H enters block B_i at $x \in C$ and exits B_i at $y \in C'$, then H' will enter G_i at $(x, 0)$ and it will exit G_i at $(y, 0)$.
3. If H exits B_i at y and enters B_{i+1} at z, then H' leaves G_i at $(y, 0)$ and goes immediately to $(z, 0)$ in G_{i+1}.

Now here is how blocks of the two types will be handled.

1. If B_i is a Type 1 block and the shorter chain has $r \geq 2$ elements, then G_i will consist of four chains in H' and they will be partitioned into two Type 1 blocks. The chain sizes will be $r + 3$ and $r + 1$ in one of them and $r + 1$ and $r - 1$ in the other.
2. If B_i is a Type 1 block and the shorter chain has a single element, G_i will consist of three chains. Two of them will have sizes 4 and 2 and will form a Type 1 block in H'. The third chain will have size 2 and is thus a Type 2 block.
3. If B_i is a Type 2 block, then G_i will consist of a Type 1 block, with one chain of size 3 and the other chain a singleton.

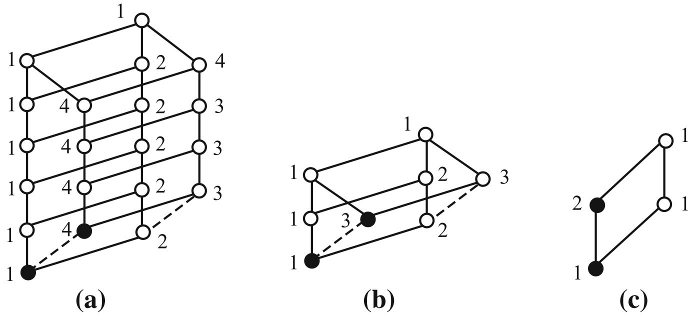


Fig. 7 **a** has large Type 1 blocks, **b** has small Type 1 blocks, and **c** has Type 2 blocks.

With these specifications in mind, the remaining details of the construction can be verified by referring to three figures:

First, in Fig. 7a, we illustrate how H' will traverse the group G_i when B_i is a Type 1 block with the smaller chain containing at least two points. The darkened points represent the entering and exiting points. The illustration has them both on the bottom, but the picture can be inverted when they are on the top. Also, which of the points is the entering point and which is the exiting point can be reversed. Regardless, we see that the four chains in H' form two Type 1 blocks. The reader should note how the extra cover required in the definition of the *strong* HC-SCP is used to move from the second chain to the third in this construction.

Second, referring to Fig. 7b, we illustrate how H' will traverse the group G_i when B_i is a Type 1 block with the smaller chain being a singleton. Here we note that one of the 2-element chains is used with a 4-element chain in forming a Type 1 block, while the remaining 2-element chain forms a Type 2 block. As above, the extra cover is essential.

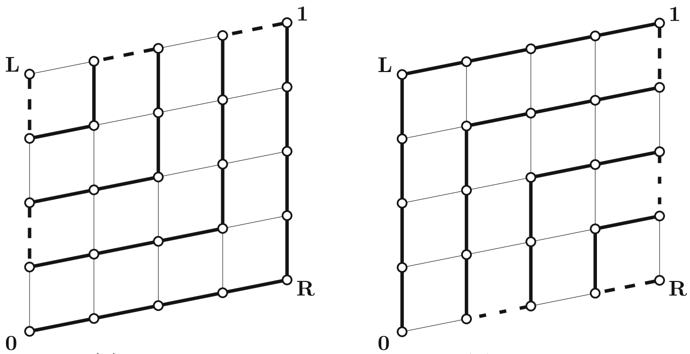
Finally, we note that the case of a Type 2 block is handled as depicted in Fig. 7c. Once the implications of the constructions detailed in these three figures has been digested, the proof of the theorem is complete. □

Now the proof of Theorem 5 follows with a trivial induction, starting with the base case $n = 1$ where the hamiltonian cycle is a single Type 2 block.

4.1 Hamiltonian Paths

We say that a leveled poset **P** satisfies the HP-SCP property if it has a hamiltonian path which parses into a symmetric chain partition. The *strong* HP-SCP property is then defined in an analogous manner. For example, when $m, p \geq 3$, the cartesian product $\mathbf{m} \times \mathbf{p}$ does not satisfy the HC-SCP property. Nevertheless, it does satisfy the strong HP-SCP property. The same argument used to show Theorem 10 also works for paths.

Corollary 11 *If* **P** *is a leveled poset satisfying the strong HP-SCP property, then so does the cartesian product* $\mathbf{P} \times \mathbf{2}$.



(a) $\mathbf{5} \times \mathbf{5}$ is $\mathfrak{L}$ **(b)** $\mathbf{5} \times \mathbf{5}$ is $\mathfrak{R}$

Fig. 8 $\mathbf{5} \times \mathbf{5}$ is $\mathfrak{B}$

5 The Strong Property is Weakly Closed

The preceding results beg the question: if **P** and **Q** each have the strong HC-SCP property, what can we say about $\mathbf{P} \times \mathbf{Q}$? In this section, we prove Theorem 6, which states that the cartesian product of any two posets with the strong HC-SCP property has the HC-SCP property. The question of whether the cartesian product has the *strong* HC-SCP property remains open (further remarks on this topic can be found in Chapter 3 of [16]).

To prove Theorem 6, we will need to show the existence of the appropriate hamiltonian cycle. We do this constructively. Because we have little control over the structures of **P** and **Q**, we will need to separate our construction into a small number of cases. The constructions will be a bit technical, but they all have the same flavor: we partition $\mathbf{P} \times \mathbf{Q}$ into small pieces, construct hamiltonian paths that parse into symmetric chains in each piece, and then show that these small paths can be put together in such a way that the result is a hamiltonian cycle in $\mathbf{P} \times \mathbf{Q}$.

5.1 Hamiltonian Paths in the Product of Chains

For positive integers m and p, we denote the points $(0, 0)$, $(0, p-1)$, $(m-1, 0)$, and $(m-1, p-1)$ of $\mathbf{m} \times \mathbf{p}$ as $\mathbf{0}$, $\mathbf{L}$, $\mathbf{R}$, and $\mathbf{1}$, respectively. We then say that $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$ if:

- $\mathbf{m} \times \mathbf{p}$ has a hamiltonian path H that parses into symmetric chains, and
- the ends of H are $\mathbf{0}$ and $\mathbf{L}$.

Similarly, we say that $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{R}$ if:

- $\mathbf{m} \times \mathbf{p}$ has a hamiltonian path H that parses into symmetric chains, and
- the ends of H are $\mathbf{0}$ and $\mathbf{R}$.

If $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$ and $\mathfrak{R}$, we call it $\mathfrak{B}$. The following facts are easily verified.

Fact 12 For positive integers m, $\mathbf{m} \times \mathbf{m}$ is $\mathfrak{B}$. For example, see Fig. 8.

Fact 13 For positive integers m and p, $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$ if and only if $\mathbf{p} \times \mathbf{m}$ is $\mathfrak{R}$.

Together with Fact 12, the following proposition implies that $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$ or $\mathfrak{R}$ for all values of m and p.

Proposition 14 *Let m and p be positive integers. If $m < p$ and m is odd, then $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$. If $m > p$ and p is odd, then $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{R}$. If $m < p$ and m is even, then $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{R}$. If $m > p$ and p is even, then $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$.*

Proof Assume $m < p$ with m odd. Consider the subposet $\mathbf{m} \times \mathbf{m}$ obtained by restricting the second coordinate to values at most $m - 1$. By Fact 12, this subposet has a hamiltonian path that witnesses the fact that $\mathbf{m} \times \mathbf{m}$ is $\mathfrak{L}$. Call this path H, and let $C_1, C_2, \ldots, C_m$ be the symmetric chains that H parses into, in the order that they appear. We now extend H to the remainder of $\mathbf{m} \times \mathbf{p}$ in the following manner: to C_i add the elements $(m-i, m), (m-i, m+1), \ldots, (m-i, p-1)$. It is easily verified that this new hamiltonian path witnesses the fact that $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$.

Now assume $p < m$ and p is odd. Applying Fact 13 to the previous case we find that $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{R}$.

The remaining two statements follow analogously. □

$\mathbf{m}\backslash\mathbf{p}$	1	2	3	4	5	6	7
1	$\mathfrak{B}$	$\mathfrak{L}$	$\mathfrak{L}$	$\mathfrak{L}$	$\mathfrak{L}$	$\mathfrak{L}$	$\mathfrak{L}$
2	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{R}$
3	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{B}$	$\mathfrak{L}$	$\mathfrak{L}$	$\mathfrak{L}$	$\mathfrak{L}$
4	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{R}$	$\mathfrak{R}$	$\mathfrak{R}$
5	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{B}$	$\mathfrak{L}$	$\mathfrak{L}$
6	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{R}$	$\mathfrak{B}$	$\mathfrak{R}$
7	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{R}$	$\mathfrak{L}$	$\mathfrak{B}$

The table above summarizes Proposition 14 for small values of m and p. The following proposition is now immediate.

Proposition 15 *If $m \geq 3$, then there exists a $\Gamma \in \{\mathfrak{L}, \mathfrak{R}\}$ such that both $\mathbf{m} \times \mathbf{p}$ and $\mathbf{m-2} \times \mathbf{p}$ are Γ. $p \geq 2$ then (m, p) is $\mathfrak{L}$.*

We are also interested in hamiltonian paths in $\mathbf{m} \times \mathbf{p}$ that use $\mathbf{1}$ as an end. To this end, we have the following definitions. We then say that $\mathbf{m} \times \mathbf{p}$ is $\overline{\mathfrak{L}}$ if:

- $\mathbf{m} \times \mathbf{p}$ has a hamiltonian path H that parses into symmetric chains, and
- the ends of H are $\mathbf{1}$ and $\mathbf{L}$.

Similarly, we say that $\mathbf{m} \times \mathbf{p}$ is $\overline{\mathfrak{R}}$ if:

- $\mathbf{m} \times \mathbf{p}$ has a hamiltonian path H that parses into symmetric chains, and
- the ends of H are $\mathbf{1}$ and $\mathbf{R}$.

The proof of the following statement is now easily verified.

Proposition 16 *If m and p are positive integers, then $\mathbf{m} \times \mathbf{p}$ is $\overline{\mathfrak{L}}$ if and only if $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{R}$, and $\mathbf{m} \times \mathbf{p}$ is $\overline{\mathfrak{R}}$ if and only if $\mathbf{m} \times \mathbf{p}$ is $\mathfrak{L}$.*

Hence, the analogous version of Proposition 15 holds as well.

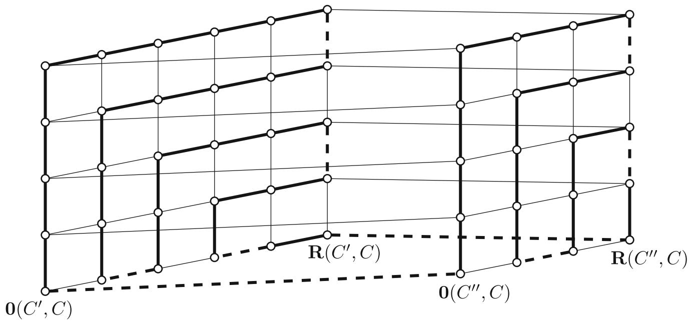


Fig. 9 The union of the *bold and dashed lines* is $\mathrm{HP}_{\mathbf{0}}(B, C)$. Here $\mathbf{m} \times \mathbf{p}$ and $\mathbf{m-2} \times \mathbf{p}$ are $\mathfrak{R}$.

5.2 Concatenating Hamiltonian Paths

In this section, we will be dealing with several cartesian products at once. Therefore, we need to refine some of our notation. If X and Y are chains, then we write $\mathbf{0}(X, Y)$, $\mathbf{L}(X, Y)$, $\mathbf{R}(X, Y)$, and $\mathbf{1}(X, Y)$ to denote the points $\mathbf{0}$, $\mathbf{L}$, $\mathbf{R}$, and $\mathbf{1}$ in $X \times Y$, respectively.

Let $\mathbf{P}$ be a poset with the strong HC-SCP property and let $B = \{C', C''\}$ be a Type 1 block in the corresponding hamiltonian cycle. Let C be a chain and $\Delta \in \{\mathbf{0}, \mathbf{L}, \mathbf{R}, \mathbf{1}\}$. Denote by $\mathrm{HP}_\Delta(B, C)$ a hamiltonian path in $B \times C$ with the following properties:

- $\mathrm{HP}_\Delta(B, C)$ parses into symmetric chains,
- the ends of $\mathrm{HP}_\Delta(B, C)$ are $\Delta(C', C)$ and $\Delta(C'', C)$, and
- the edge with ends $\Delta(C', C)$ and $\Delta(C'', C)$ is not in $\mathrm{HP}_\Delta(B, C)$.

For an example, see Fig. 9.

Proposition 17 *Let $B = \{C', C''\}$ be a Type 1 block and let C be a chain. Then $HP_\Delta(B, C)$ exists for each $\Delta \in \{\mathbf{0}, \mathbf{L}, \mathbf{R}, \mathbf{1}\}$.*

Proof Let $\max\{|C'|, |C''|\} = m$ and $|C| = p$. Clearly $m \geq 3$. By Proposition 15, there is a $\Gamma \in \{\mathfrak{L}, \mathfrak{R}\}$ such that both $\mathbf{m} \times \mathbf{p}$ and $\mathbf{m-2} \times \mathbf{p}$ are Γ. We may assume $\Gamma = \mathfrak{L}$, as the other case follows analogously.

Let H' and H'' be the hamiltonian paths in $C' \times C$ and $C'' \times C$, respectively, that certify that each is $\mathfrak{L}$. To find $\mathrm{HP}_{\mathbf{0}}(B, C)$, traverse H' from $\mathbf{0}(C', C)$ to $\mathbf{L}(C', C)$, follow the edge from $\mathbf{L}(C', C)$ to $\mathbf{L}(C'', C)$, then traverse H'' from $\mathbf{L}(C'', C)$ to $\mathbf{0}(C'', C)$. To find $\mathrm{HP}_{\mathbf{L}}(B, C)$, traverse H' from $\mathbf{L}(C', C)$ to $\mathbf{0}(C', C)$, follow the edge from $\mathbf{0}(C', C)$ to $\mathbf{0}(C'', C)$, then traverse H'' from $\mathbf{0}(C'', C)$ to $\mathbf{L}(C'', C)$.

Proposition 16 implies that $C' \times C$ and $C'' \times C$ are also $\overline{\mathfrak{R}}$. In order to find $\mathrm{HP}_{\mathbf{R}}(B, C)$ and $\mathrm{HP}_{\mathbf{1}}(B, C)$, we proceed analogously to the previous case, replacing each $\mathbf{0}$ with $\mathbf{R}$ and each $\mathbf{L}$ with $\mathbf{1}$. □

Suppose $\mathbf{P}$ has the strong HC-SCP property and let H be the hamiltonian cycle that witnesses this. We divide the Type 1 and Type 2 blocks that partition the chains in H into two subclasses. When $B = \{C_i, C_{i+1}\}$ be a Type 1 block, where C_i comes before C_{i+1} in H, we call B *up-down*, or UD, if H traverses C_i from the least element to the

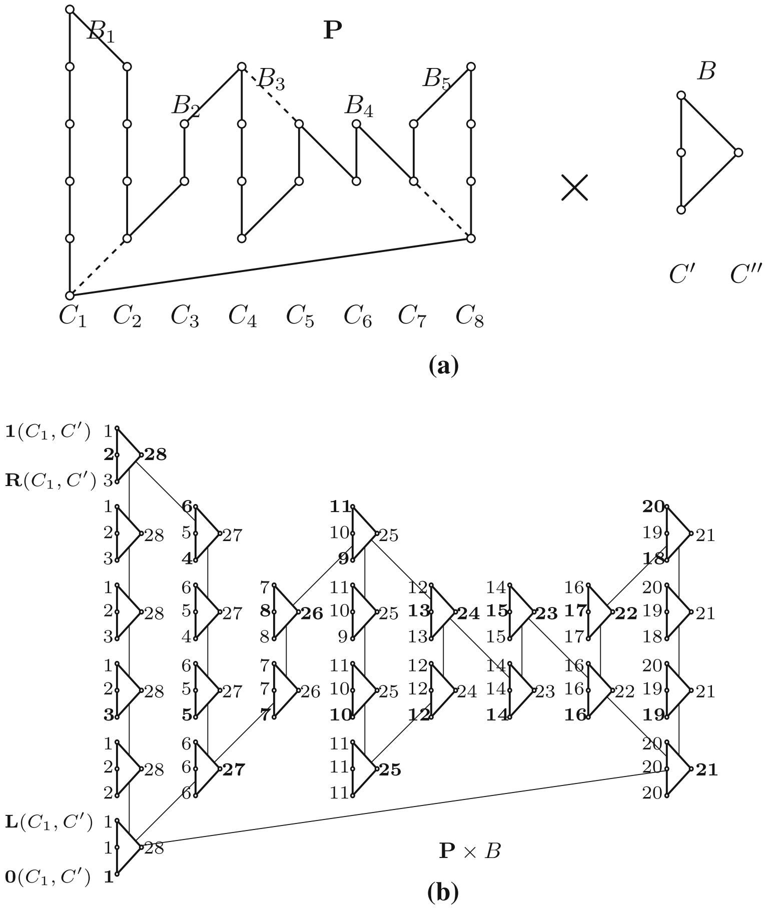


Fig. 10 The numbers in **b** are the symmetric chains. The first occurrence of a number in the hamiltonian cycle is shown in bold

greatest element (and hence traverses C_{i+1} from greatest element to least element). Otherwise we call B *down-up*, or DU. When B is Type 2, we call B *up*, or U, if H traverses B from the least element to the greatest element. Otherwise we call B *down*, or D. For an example, see Fig. 10. The block types there are: B_1 and B_5 are UD; B_2 and B_4 are U; B_3 is DU.

Now let $\mathbf{P}$ be a poset with the strong HC-SCP property that is witnessed by the hamiltonian cycle H with chains $C_1, C_2, \ldots, C_w$. Also let $B = \{C', C''\}$ be a Type 1 block and let $\Delta \in \{\mathbf{0}, \mathbf{L}, \mathbf{R}, \mathbf{1}\}$. Denote by $\mathrm{HP}_\Delta(\mathbf{P}, B)$ a hamiltonian path in $\mathbf{P} \times B$ with the following properties:

- $\mathrm{HP}_\Delta(\mathbf{P}, B)$ parses into symmetric chains,
- the ends of $\mathrm{HP}_\Delta(\mathbf{P}, B)$ are $\Delta(C_1, C')$ and $\Delta(C_1, C'')$, and
- the edge with ends $\Delta(C_1, C')$ and $\Delta(C_1, C'')$ is not in $\mathrm{HP}_\Delta(\mathbf{P}, B)$.

For an example of the following construction, see Fig. 10.

Proposition 18 *Suppose* $\mathbf{P}$ *is a poset with the strong HC-SCP property that is witnessed by the hamiltonian cycle H with chains* $C_1, C_2, \ldots, C_w$. *Also let* $B = \{C', C''\}$ *be a Type 1 block. If* C_1 *is in a UD or U block, then* $HP_\Delta(\mathbf{P}, B)$ *exists for each* $\Delta \in \{\mathbf{0}, \mathbf{L}\}$. *Otherwise* $HP_\Delta(\mathbf{P}, B)$ *exists for each* $\Delta \in \{\mathbf{R}, \mathbf{1}\}$.

Proof Let $B_1, B_2, \ldots, B_s$ be the blocks that partition the chains in H. We may assume that B_1 is UD or U and prove that $\mathrm{HP}_\Delta(\mathbf{P}, B)$ exists for each $\Delta \in \{\mathbf{0}, \mathbf{L}\}$, as otherwise we the same proof on the dual of $\mathbf{P}$.

We construct $\mathrm{HP}_\mathbf{0}(\mathbf{P}, B)$ by visiting vertices in the following order: $B_1 \times C'$, $B_2 \times C'$, $\ldots$, $B_s \times C'$, $B_s \times C''$, $B_{s-1} \times C''$, $\ldots$, $B_1 \times C''$. The manner in which we visit the vertices in the individual cartesian products depends on B_i. First suppose B_i is Type 1. If $B_i = \{C_i, C_{i+1}\}$ is UD, then we traverse $B_i \times C'$ with $\mathrm{HP}_\mathbf{0}(B_i, C')$ from $\mathbf{0}(C_i, C')$ to $\mathbf{0}(C_{i+1}, C')$, and we traverse $B_i \times C''$ with $\mathrm{HP}_\mathbf{0}(B_i, C'')$ from $\mathbf{0}(C_{i+1}, C'')$ to $\mathbf{0}(C_i, C'')$. If B_i is DU, then we traverse $B_i \times C'$ with $\mathrm{HP}_\mathbf{R}(B_i, C')$ from $\mathbf{R}(C_i, C')$ to $\mathbf{R}(C_{i+1}, C')$, and we traverse $B_i \times C''$ with $\mathrm{HP}_\mathbf{R}(B_i, C'')$ from $\mathbf{R}(C_{i+1}, C'')$ to $\mathbf{R}(C_i, C'')$.

Next suppose B_i is Type 2. We know from Proposition 14 that $\mathbf{2} \times C$ is $\mathfrak{R}$ for any chain C. Let $H_{C'}$ and $H_{C''}$ be the hamiltonian paths that witness this for $B_i \times C'$ and $B_i \times C''$, respectively. For our construction, if B_i is U, then we traverse $B_i \times C'$ from $\mathbf{0}(B_i, C')$ to $\mathbf{R}(B_i, C')$ with $H_{C'}$, and we traverse $B_i \times C''$ from $\mathbf{R}(B_i, C'')$ to $\mathbf{0}(B_i, C'')$ with $H_{C''}$. If B_i is D, then we traverse $B_i \times C'$ from $\mathbf{R}(B_i, C')$ to $\mathbf{0}(B_i, C')$ with $H_{C'}$, and we traverse $B_i \times C''$ from $\mathbf{0}(B_i, C'')$ to $\mathbf{R}(B_i, C'')$ with $H_{C''}$.

To this point, we have partitioned the vertices of $\mathbf{P} \times B$ into symmetric chains, and further, we have described a sequence of paths that starts at $\mathbf{0}(C_1, C')$, ends at $\mathbf{0}(C_1, C'')$, and avoids the edge from $\mathbf{0}(C_1, C')$ to $\mathbf{0}(C_1, C'')$. It remains to check that the paths we have described can be linked with edges in the cover graph of $\mathbf{P} \times B$ to create a hamiltonian path. To this end, the following observations are sufficient:

- Suppose the traversal of $B_i \times C'$ ends at (x_1, y_1) and the traversal of $B_{i+1} \times C'$ starts (x_2, y_2), for $i \in [s-1]$. Then x_1x_2 is an edge in H.
- Suppose the traversal of $B_{i+1} \times C''$ ends at (x_1, y_1) and the traversal of $B_i \times C''$ starts at (x_2, y_2), for $i \in [s-1]$. Then x_1x_2 is an edge in H.
- Suppose the traversal of $B_s \times C'$ ends at (x_1, y_1) and the traversal of $B_s \times C''$ starts at (x_2, y_2). Then y_1y_2 is an edge in the cover graph of B.

We leave the verification of these statements to the reader.

We construct $\mathrm{HP}_\mathbf{L}(\mathbf{P}, B)$ by visiting the cartesian products in the same order. However, the manner in which we visit the vertices in the individual cartesian products is different. As these details are completely analogous to the previous case, we leave them to the reader, noting that in order to complete the proof one replaces $\mathbf{0}$ by $\mathbf{L}$, $\mathbf{R}$ by $\mathbf{1}$, and $\mathfrak{R}$ by $\overline{\mathfrak{L}}$. □

Let $\mathbf{P}_i$ be a poset with the strong HC-SCP property that is witnessed by the hamiltonian cycles H_i with chains $C_1^i, C_2^i, \ldots, C_{w_i}^i$ for $i \in \{1, 2\}$. Let $\Delta \in \{\mathbf{0}, \mathbf{L}, \mathbf{R}, \mathbf{1}\}$. Denote by $\mathrm{HP}_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ a hamiltonian path in $\mathbf{P}_1 \times \mathbf{P}_2$ with the following properties:

- $\mathrm{HP}_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ parses into symmetric chains,
- the ends of $\mathrm{HP}_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ are $\Delta(C_1^1, C_1^2)$ and $\Delta(C_1^1, C_{w_2}^2)$, and

- the edge with ends $\Delta(C^1_1, C^2_1)$ and $\Delta(C^1_1, C^2_{w_2})$ is not in $\text{HP}_\Delta(\mathbf{P}_1, \mathbf{P}_2)$.

We can now prove the main result of this subsection.

Proposition 19 *Suppose* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are posets with the strong HC-SCP property, which are witnessed by* H_1 *and* H_2*, respectively. Further suppose that all of the blocks in* H_2 *are Type 1. If* C^1_1 *is in a UD or U block, then* $HP_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ *exists for each* $\Delta \in \{\mathbf{0}, \mathbf{L}\}$*. Otherwise* $HP_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ *exists for each* $\Delta \in \{\mathbf{R}, \mathbf{1}\}$*. In particular,* $\mathbf{P}_1 \times \mathbf{P}_2$ *has a hamiltonian cycle that parses into symmetric chains.*

Proof We may assume that the block that contains C^1_1 is in a UD or U and prove $\text{HP}_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ exists for $\Delta \in \{\mathbf{0}, \mathbf{L}\}$, as else we apply the same proof using the dual of $\mathbf{P}_1$.

Let $B_i = \{C^2_{2i-1}, C^2_{2i}\}$ for $i \in [w_2/2]$ be the Type 1 blocks in H_2, and let $\Delta \in \{\mathbf{0}, \mathbf{L}\}$. We construct $\text{HP}_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ by traversing the vertices in the following order: $\text{HP}_\Delta(\mathbf{P}_1, B_1)$, $\text{HP}_\Delta(\mathbf{P}_1, B_2), \ldots, \text{HP}_\Delta(\mathbf{P}_1, B_{w_2/2})$, all of which exist by Proposition 18. Clearly every vertex of $\mathbf{P}_1 \times \mathbf{P}_2$ is used in exactly one of these paths, and every chain used is symmetric. We can link these paths together by noticing that the ends of each path are in the same copy of $\mathbf{P}_2$. In particular, we link these paths with the edges that join $\Delta(C^1_1, C^2_{2i})$ and $\Delta(C^1_1, C^2_{2i+1})$ for all $i \in [w_2/2 - 1]$.

Since $\text{HP}_\Delta(\mathbf{P}_1, \mathbf{P}_2)$ avoids the edge that joins $\Delta(C^1_1, C^2_1)$ and $\Delta(C^1_1, C^2_{w_2})$, we add this edge to complete the desired hamiltonian cycle. □

Our goal in the next two sections will be to get rid of the extra condition in Proposition 19 that $\mathbf{P}_2$ can be partitioned into Type 1 blocks only. As it turns out, the method we will use depends on the width of $\mathbf{P}_2$.

6 Even Width

If either poset in our cartesian product has a hamiltonian cycle that parses into symmetric chains such that every block is Type 1, then we are done by Proposition 19, using that poset as $\mathbf{P}_2$. So in this section we assume that our posets do not have such a block structure. This allows us the following facts, which are easily verified.

Fact 20 Let $\mathbf{P}$ be a poset with the strong HC-SCP property and suppose this is witnessed by a hamiltonian cycle with a Type 2 block. Then the height of $\mathbf{P}$ is even.

Fact 21 Let $\mathbf{P}$ be a poset with the strong HC-SCP property and suppose this is witnessed by a hamiltonian cycle with a Type 2 block. Then all Type 2 blocks have the same type; they are either all U or all D.

Let $k \geq 2$ be a positive integer. Let $\mathbf{P}$ be a poset with ground set $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$ and partial order consisting of comparabilities $a_i < b_i$ for each $i \in [k]$ and

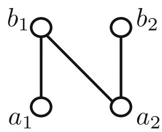


Fig. 11 The poset **N**

$a_{i+1} < b_i$ for each $i \in [k-1]$. Then we call $\mathbf{P}$ a *fence of length* k, denoted $\mathbb{F}_k$. The fence of length two is also called $\mathbf{N}$, and is depicted in Fig. 11. If we add the comparability $a_1 < b_k$ to $\mathbb{F}_k$ we get the *crown of length* k, denoted $\mathbb{C}_k$.

We introduce the next definitions to simplify our exposition. Let $\mathbf{P}$ be a poset with the strong HC-SCP property, witnessed by hamiltonian cycle H with chains $C_1, C_2, \ldots, C_w$. We say $\mathbf{P}$ is in *standard position* when C_w has type U. When $\mathbf{P}$ is in *standard position*, we denote the least element of C_w by w_1, the greatest by w_2, and the element of C_1 that is adjacent to w_2 in H by s. Notice that, if $\mathbf{P}$ is in *standard position*, C_1 is either a 2-element chain in a UD or U block or it is a 4-element chain in a DU block.

The next fact follows from a careful reading of the proofs in Sect. 4.

Fact 22 Let $\mathbf{P}$ be a poset with the strong HC-SCP property which is witnessed by a hamiltonian cycle with chains $C_1, C_2, \ldots, C_w$ and $|C_w| = 2$. Then $\mathbf{P} \times \mathbf{2}$ has two distinct hamiltonian paths that parse into symmetric chains and can be extended to hamiltonian cycles. If C_1 is in a UD or U block, then one such path has ends $\mathbf{0}(C_1, \mathbf{2})$ and $\mathbf{R}(C_w, \mathbf{2})$ (this path is constructed in Sect. 4), and the other uses the same chains as the first (in slightly different order) and has ends $\mathbf{L}(C_1, \mathbf{2})$ and $\mathbf{1}(C_w, \mathbf{2})$.

If C_1 is in a DU or D block, then one of the paths has ends $\mathbf{R}(C_1, \mathbf{2})$ and $\mathbf{R}(C_w, \mathbf{2})$, and the other uses the same chains as the first (in slightly different order) and has ends $\mathbf{1}(C_1, \mathbf{2})$ and $\mathbf{1}(C_w, \mathbf{2})$.

Lemma 23 *Let* $\mathbf{P}$ *be a poset with the strong HC-SCP property, witnessed by a hamiltonian cycle* H. *Suppose* $\mathbf{P}$ *is in standard position. Then* $\mathbf{P} \times \mathbf{N}$ *has a hamiltonian path that parses into symmetric chains with ends* (s, a_1) *and* (s, b_2).

Proof Let $\mathcal{C}_i$ be the subposet of $\mathbf{N}$ induced by a_i and b_i, for $i \in \{1, 2\}$, and let $C_1, C_2, \ldots, C_w$ be the chains of H. First suppose that C_1 is in a UD or U block. According to Fact 22, there is a hamiltonian path, H_1, in $\mathbf{P} \times \mathcal{C}_1$ that parses into symmetric chains, starts at $\mathbf{0}(C_1, \mathcal{C}_1) = (s, a_1)$, and ends at $\mathbf{R}(C_w, \mathcal{C}_1) = (w_2, a_1)$. However, since $\mathbf{2} \times \mathbf{2}$ is both $\mathfrak{L}$ and $\mathfrak{R}$, we can amend H_1 so that it ends at (w_1, b_1) instead; the last two chains are $\{(w_1, a_1), (w_2, a_1), (w_2, b_1)\}$ and $\{(w_1, b_1)\}$. Call this amended hamiltonian path H_1'.

Using Fact 22 again, we know there is a hamiltonian path, H_2, in $\mathbf{P} \times \mathcal{C}_2$ that parses into symmetric chains, starts at $\mathbf{L}(C_1, \mathcal{C}_2) = (s, b_2)$, and ends at $\mathbf{1}(C_w, \mathcal{C}_2) = (w_2, b_2)$. Similarly, we can amend this path to end at (w_1, a_2) instead. Call this amended path H_2'. Now we find the desired path in $\mathbf{P} \times \mathbf{N}$ by traversing H_1', following the edge from (w_1, b_1) to (w_1, a_2), then traversing H_2' in reverse.

Second, suppose that C_1 is in a DU block. The analogous proof holds, where the changes are: H_1 has ends $\mathbf{R}(C_1, \mathcal{C}_1) = (s, a_1)$ and $\mathbf{R}(C_w, \mathcal{C}_1) = (w_2, a_1)$, and H_2 has ends $\mathbf{1}(C_1, \mathcal{C}_2) = (s, b_2)$ and $\mathbf{1}(C_w, \mathcal{C}_2) = (w_2, b_2)$. □

The following lemma is an immediate consequence of Lemma 23, noting that $\mathbb{F}_{2k}$ is simply k $\mathbf{N}$'s concatenated together.

Lemma 24 *Let* k *be a positive integer and let* $\mathbf{P}$ *be a poset with the strong HC-SCP property. When* $\mathbf{P}$ *is in standard position*, $\mathbf{P} \times \mathbb{F}_{2k}$ *has a hamiltonian path that parses into symmetric chains with ends* (s, a_1) *and* (s, b_{2k}).

In what follows, we denote the hamiltonian path constructed in Lemma 24 by $\text{HP}(\mathbf{P}, \mathbb{F}_{2k})$, of simply $\text{HP}(\mathbf{P}, \mathbf{N})$ when $k = 1$.

The purpose behind the preceding work will now become clear; 2-element chains, seen together as fences, serve as the basis of our general construction. Let $\mathbf{P}$ be a poset of even width with the strong HC-SCP property that is witnessed by hamiltonian cycle H, and let $C_1, C_2, \ldots, C_w$ be the chains in H. Suppose $C_{i_1}, C_{i_2}, \ldots, C_{i_{2k}}$ is the subsequence of Type 2 chains. Let $\mathbf{N}_j$ denote the pair $\{C_{i_{2j-1}}, C_{i_{2j}}\}$ for $j \in \{1, 2 \ldots, k\}$. Define a *group of Type 1 blocks* to be a maximal sequence of consecutive chains of H such that each chain is part of a Type 1 block. These groups can arise in two different forms. We say a group G is *inside* if there is some j such that $C_{i_{2j-1}}$ and $C_{i_{2j}}$ are the chains in H that come immediately before and immediately after the chains of G, respectively. Otherwise we say G is *outside*. The group before C_{i_1} and the group after C_{i_k} are considered as distinct outside groups.

Theorem 25 *Suppose* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are posets with the strong HC-SCP property. Further suppose that the width of* $\mathbf{P}_2$ *is even. Then* $\mathbf{P}_1 \times \mathbf{P}_2$ *has the HC-SCP property.*

Proof Let H_1 and H_2 be hamiltonian cycles that witness the fact that $\mathbf{P}_1$ and $\mathbf{P}_2$ have the strong HC-SCP property, respectively. By Proposition 19, we may assume H_1 and H_2 have blocks of Type 2. Orient $\mathbf{P}_1$ so that it is in *standard position*; that is, let $D_1, D_2, \ldots, D_w$ be the chains of H_1 with $D_w = \{w_1, w_2\}$ a U block. Orient $\mathbf{P}_2$ so the Type 2 blocks in H_2 are U.

Suppose initially that H_2 has only Type 2 blocks. Then $\mathbf{P}_2$ is isomorphic to $\mathbb{C}_{2k}$ for some positive integer k. Lemma 24 allows us to find $\text{HP}(\mathbf{P}_1, \mathbb{F}_{2k})$, to which we add the edge from (s, a_1) to (s, b_{2k}) to find the desired hamiltonian cycle.

So we may assume H_2 has at least one Type 1 block. Let $C_{i_1}, C_{i_2}, \ldots, C_{i_{2k}}$ be the subsequence of Type 2 chains in H_2, and let $\mathbf{N}_j$ denote the pair $\{C_{i_{2j-1}}, C_{i_{2j}}\}$ for all $j \in [k]$. We can now piece together the desired hamiltonian cycle. Suppose that we are either starting our construction or that we have some nonempty part of our cycle constructed. In the latter case, suppose we have visited exactly the points in $\mathbf{P}_1 \times \{C_1 \cup C_2 \cup \ldots C_t\}$, where C_t is either the second Type 2 block of some $\mathbf{N}_j$ or the last chain in some outside group, and have ended in $\{s\} \times \mathbf{P}_2$. We then proceed as follows.

There are three cases to consider. First, suppose C_{t+1} is the first Type 2 block in $\mathbf{N}_{j+1}$, and suppose $\mathbf{N}_{j+1}$ does not have a group of Type 1 blocks inside of it. Then we add $\text{HP}(\mathbf{P}_1, \mathbf{N}_{j+1})$ to our cycle.

Second, suppose C_{t+1} is the first Type 2 block in $\mathbf{N}_{j+1}$ and G is a group of Type 1 blocks inside of it. Let $\mathbf{N}_{j+1}$ consist of the points a_1, b_1, a_2, and b_2, as in Fig. 11, and let the blocks of G be $B_1, B_2, \ldots, B_q$. We now add points to our cycle in the following way: start by adding the first half of $\text{HP}(\mathbf{P}_1, \mathbf{N}_{j+1})$ via the construction in Lemma 23 (there it was called H_1'). Follow the edge from (w_1, b_1) to a vertex in (w_1, B_1) via H_2. Next add $\text{HP}_\Delta(\mathbf{P}_1, G)$, where $\Delta = \mathbf{0}$ if B_1 is UD and $\Delta = \mathbf{L}$ if B_1 is DU. (Notice first that $\text{HP}_\Delta(\mathbf{P}_1, G)$ is a slight abuse of notation, although we feel confident that its meaning is clear. Second, notice that each $\text{HP}_\Delta(\mathbf{P}_1, B_i)$ in $\text{HP}_\Delta(\mathbf{P}_1, G)$ traverses the chains of H_1 in the order $D_w, D_1, D_2, \ldots, D_{w-1}$, and back.) Next, follow the edge in H_2 to (w_1, a_2). Finally, add the second half of HP($\mathbf{P}_1, \mathbf{N}_{j+1}$) via the construction in Lemma 23 (there is was called H_2') in reverse.

Last, suppose C_{t+1} is the first chain in an outside group of Type 1 blocks, G. Then we add $\text{HP}_\Delta(\mathbf{P}_1, G)$, where again $\Delta \in \{\mathbf{0}, \mathbf{L}\}$ and depends on the orientation of the first block in G.

In each case we return to the initial conditions; we have visited exactly the points in $\mathbf{P}_1 \times \{C_1 \cup C_2 \cup \ldots C_{t'}\}$, where $C_{t'}$ is either the second Type 2 block of some $\mathbf{N}_j$ or the last chain in some outside group, and have ended in $\{s\} \times \mathbf{P}_2$. Therefore we can continue this process until all of the points in $\mathbf{P}_1 \times \mathbf{P}_2$ have been visited.

We now have a partition of the points in $\mathbf{P}_1 \times \mathbf{P}_2$ into symmetric chains. It remains to show that we can link the hamiltonian paths produced in the construction above in order to obtain a hamiltonian cycle. We can do this with edges in H_2, as all such transitions occur in $\{s\} \times \mathbf{P}_2$. □

An example of this construction can be found in Fig. 13, where it is combined with the techniques described in the next section.

7 Odd Width

Just as we started with even fences and even crowns in the previous section, we start with odd fences and odd crowns here.

Lemma 26 *Let j be a positive integer. Then $\mathbb{F}_3 \times \mathbb{F}_{2j+1}$ has a hamiltonian path that parses into symmetric chains with ends (a_1, a_1) and (b_3, a_1).*

Proof We give an explicit construction of the chains that are used. First, we cover $\{a_1, b_1, a_2, b_2\} \times \{a_1, b_1, a_2, b_2\}$, which is isomorphic to $\mathbf{N} \times \mathbf{N}$, by the eight chains

$$\{(a_1, a_1), (a_1, b_1), (b_1, b_1)\},\ \ \{(b_1, a_1)\},\ \ \{(a_2, a_1), (b_2, a_1), (b_2, b_1)\},\ \ \{(a_2, b_1)\},$$
$$\{(a_2, a_2), (b_2, a_2), (b_2, b_2)\},\ \ \{(a_2, b_2)\},\ \ \{(b_1, b_2), (b_1, a_2), (a_1, a_2)\},\ \ \{(a_1, b_2)\}.$$

We then cover all elements in $\{a_1, b_1, a_2, b_2\} \times \{a_3, b_3, a_4, b_4, \ldots, a_{2j}, b_{2j}\}$ using this same strategy, ending at the point (a_1, b_{2j}). We cover the remaining elements in $\{a_1, b_1, a_2, b_2\} \times \mathbb{F}_{2j+1}$ with the four chains

$$\{(a_1, a_{2j+1}), (a_1, b_{2j+1}), (b_1, b_{2j+1})\},\ \ \{(b_1, a_{2j+1})\},$$
$$\{(a_2, a_{2j+1}), (a_2, b_{2j+1}), (b_2, b_{2j+1})\},\ \ \{(b_2, a_{2j+1})\}.$$

Next we use the chains $\{(a_3, a_{2j+1}), (a_3, b_{2j+1}), (b_3, b_{2j+1})\}$ and $\{(b_3, a_{2j+1})\}$. Finally, we use the chains $\{(b_3, b_i), (a_3, b_i), (a_3, a_i)\}$ and $\{(b_3, a_i)\}$ for $i = 2j, 2j - 1, \ldots, 1$, in that order, to finish the desired hamiltonian path. □

The following corollary is an immediate consequence of Lemma 26 and Lemma 23, noting that $\mathbb{F}_{2k+1}$ is simply $\mathbb{F}_3$ followed by $k - 1$ $\mathbf{N}$'s concatenated together.

Lemma 27 *Let k and j be positive integers. Then $\mathbb{F}_{2k+1} \times \mathbb{F}_{2j+1}$ has a hamiltonian path that parses into symmetric chains with ends (a_1, a_1) and (b_{2k+1}, a_1).*

Before proving the main result of this section, we require two technical lemmas concerning Type 1 blocks with chains of length 2 and 4.

Lemma 28 *Let* **P** *be a poset with the strong HC-SCP property that is witnessed by hamiltonian cycle H, and suppose H has a Type 2 block. Let $G = \{B_1, B_2, \ldots, B_s\}$ be a maximal consecutive set of Type 1 blocks of H. Then there is a $B_i = \{C_1^i, C_2^i\} \in G$ such that $\{|C_1^i|, |C_2^i|\} = \{2, 4\}$.*

Proof Let B_0 and B_{s+1} be the Type 2 blocks before and after G, where $B_0 = B_{s+1}$ if H has just one Type 2 block. We may assume $|C_1^1| = |C_2^s| = 4$ and $|C_2^1| = |C_1^s| = 6$, as otherwise we are done. In other words, $(|C_1^1|, |C_2^1|) \equiv (0, 2) \mod 4$ and $(|C_1^s|, |C_2^s|) \equiv (2, 0) \mod 4$. But this switch can only occur if there is some $i \in \{2, 3, \ldots, s-1\}$ with $\{|C_1^i|, |C_2^i|\} = \{2, 4\}$. □

Lemma 29 *Let* **P** *be a poset with a hamiltonian cycle H that parses into symmetric chains $C_1, C_2, \ldots, C_t$, with t odd, and* **P** *in standard position. Let $B = \{C', C''\}$ be a Type 1 block such that $C' = x_1 < x_2 < x_3 < x_4$ and $C'' = y_1 < y_2$. Then $\mathbf{P} \times B$ has a hamiltonian path H' that parses into symmetric chains, starts at α and ends at β, for each pair $(\alpha, \beta) \in \{((w_1, x_4), (s, y_2)), ((w_1, x_1), (s, y_1)), ((w_1, y_2), (s, x_4)), ((w_1, y_1), (s, x_1))\}$.*

Proof We prove the Lemma only for $(\alpha, \beta) = ((w_1, x_4), (s, y_2))$, as the other three cases follow analogously. Now, since **P** is in *standard position*, we know that C_t is U. We shall refer to the block containing C_1 as B_1, and refer to the points in C_i as $c_{i,1} < c_{i,2} < \cdots < c_{i,k}$ (thus $w_1 = c_{t,1}$ and $w_2 = c_{t,2}$). We proceed by proving two claims and then by showing that these claims imply the Lemma.

Claim 30 The Lemma holds when all blocks in H are Type 1 except for C_t. In particular, if B_1 is UD or U, then H' starts at $(c_{t,1}, x_4)$ and ends at $(c_{1,1}, y_2)$, and if B_1 is DU, then H', starts at $(c_{t,1}, x_4)$ and ends at $(c_{1,k}, y_2)$.

We prove Claim 30 using induction on t. If $t = 1$, in which case $C_1 = C_t$, then H' has the chains $\{(c_{1,1}, x_4), (c_{1,1}, x_3), (c_{1,1}, x_2)\}$, $\{(c_{1,1}, x_1), (c_{1,2}, x_1), (c_{1,2}, x_2), (c_{1,2}, x_3), (c_{1,2}, x_4)\}$, $\{(c_{1,2}, y_2), (c_{1,2}, y_1), (c_{1,1}, y_1)\}$, and $\{(c_{1,1}, y_2)\}$.

Now assume $t \geq 3$. First, suppose B_1 is UD. Therefore, the block containing C_3 is also UD or U, $|C_1| = 2$, and $|C_2| = 4$. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from $(c_{t,1}, x_4)$ to $(c_{3,1}, y_2)$. We complete H' in the following manner: use the edge from $(c_{3,1}, y_2)$ to $(c_{2,1}, y_2)$, traverse $\mathrm{HP}_{\mathbf{R}}(B, C_2)$ to $(c_{2,1}, x_4)$, use the edge from $(c_{2,1}, x_4)$ to $(c_{1,1}, x_4)$, and traverse $\mathrm{HP}_{\mathbf{R}}(B, C_1)$ to $(c_{1,1}, y_2)$.

Second, suppose B_1 is DU. Then $|C_1| = 4$ and $|C_2| \in \{2, 6\}$. Assume the block containing C_3 is DU. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from $(c_{t,1}, x_4)$ to $(c_{3,k}, y_2)$. We complete H' in the following manner: use the edge from $(c_{3,k}, y_2)$ to $(c_{2,k}, y_2)$, traverse $\mathrm{HP}_{\mathbf{1}}(B, C_2)$ to $(c_{2,k}, x_4)$, use the edge from $(c_{2,k}, x_4)$ to $(c_{1,k}, x_4)$, and traverse $\mathrm{HP}_{\mathbf{1}}(B, C_1)$ to $(c_{1,k}, y_2)$. Now assume the block containing C_3 is UD or U. Then $|C_2| = 2$. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from $(c_{t,1}, x_4)$ to $(c_{3,1}, y_2)$. We complete H' by using the edge from

$(c_{3,1}, y_2)$ to $(c_{2,k}, y_2)$ and then by following the same steps as in the preceding case.

Claim 31 Suppose all blocks in $\mathbf{P}$ are Type 1 except C_t. If B_1 is UD or U, then there is a hamiltonian path that parses into symmetric chains, starts at $(c_{t,2}, y_2)$, and ends at $(c_{1,1}, x_1)$. If B_1 is DU, then H'' starts at $(c_{t,2}, y_2)$ and ends at $(c_{1,k}, x_1)$.

We prove Claim 31 using induction on t. If $t = 1$, in which case $C_1 = C_t$, then H'' has the chains $\{(c_{1,2}, y_2), (c_{1,1}, y_2), (c_{1,1}, y_1)\}$, $\{(c_{1,2}, y_1)\}$, $\{(c_{1,2}, x_1), (c_{1,2}, x_2), (c_{1,2}, x_3)\}$, and $\{(c_{1,2}, x_4), (c_{1,1}, x_4), (c_{1,1}, x_3), (c_{1,1}, x_2), (c_{1,1}, x_1)\}$.

The remaining details are completely analogous to those of Claim 30, so we leave them to the reader, noting that we replace $\mathbf{R}$ by $\mathbf{0}$ when B_1 is UD and $\mathbf{1}$ by $\mathbf{L}$ when B_1 is DU.

We can now complete the proof of Lemma 29. Since t is odd there are an odd number of Type 2 blocks in H. Label these Type 2 blocks $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{2j-1} = C_t$. Define $\mathcal{B}_1$ to be the elements in H prior to $\mathcal{C}_1$, and define $\mathcal{B}_i$ to be the set of elements in Type 1 blocks between $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ for $i \geq 2$. Then define $\mathcal{D}_i = \mathcal{C}_i \cup \mathcal{B}_i$. Notice that the $\mathcal{D}_i$ partition the elements of $\mathbf{P}$.

We now construct H' as follows: for odd i we traverse $\mathcal{D}_i \times B$ using the method of Claim 30, and for even i we traverse $\mathcal{D}_i \times B$ using the method of Claim 31. We need to verify that these traversals can be linked by edges in $\mathbf{P} \times B$. Going from $\mathcal{D}_{2i+1}$ to $\mathcal{D}_{2i}$ for $i \in \{2, 3, \ldots, j\}$ is straightforward given the statements of the claims. Going from $\mathcal{D}_{2i}$ to $\mathcal{D}_{2i-1}$ for $i \in [j-1]$ needs a quick justification since the traversal of $\mathcal{C}_{2i-1} \times B$ for $i \in [j-1]$ must behave differently than the traversal of $C_t \times B$. Let $\mathcal{C}_{2i-1}$ be the chain $z_1 < z_2$. Then we traverse $\mathcal{C}_{2i-1} \times B$ as follows: from $\mathcal{D}_{2i}$ we follow the edge to (z_2, x_1), then we use the chains $\{(z_2, x_1), (z_2, x_2), (z_3, x_3)\}$, $\{(z_2, x_4), (z_1, x_4), (z_1, x_3), (z_1, x_2), (z_1, x_1)\}$, $\{(z_1, y_1), (z_2, y_1), (z_2, y_2)\}$, $\{(z_1, y_2)\}$. Now we can continue our path through $\mathcal{D}_{2i-1}$ as before.

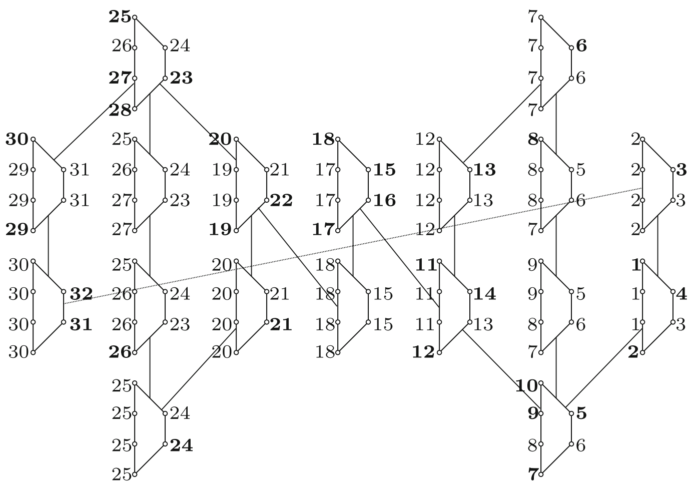


Fig. 12 An example of $\mathbf{P} \times B$ from the proof of Lemma 29

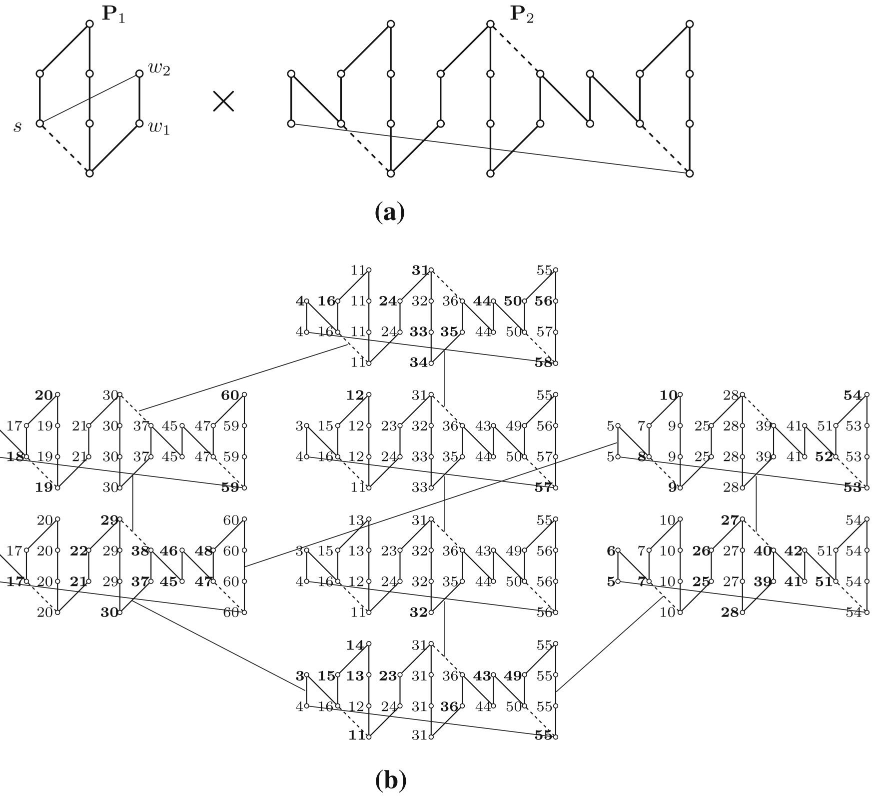


Fig. 13 **a** $\mathbf{P}_1$ is in standard position. The first three chains of $\mathbf{P}_2$ constitute $\mathbf{P}'$ in the proof of Theorem 32. The second and third chains of $\mathbf{P}'' = \mathbf{P_2} - \mathbf{P}'$ are an inside group, and the last two chains of $\mathbf{P}''$ are an outside group. **b** The numbers are the symmetric chains of $\mathbf{P_1} \times \mathbf{P_2}$. The first occurrence of a number in the hamiltonian cycle is shown in bold

Since we have an odd number of $\mathcal{D}_i$, the last element used in H' is (s, y_2), as desired. See Fig. 12 for an example. □

When taken together with Theorem 25, Theorem 32 finishes the proof of Theorem 6. For a visual representation of the details in Theorem 32, see Fig. 13. In what follows, the hamiltonian path constructed in Lemma 27 will be denoted $\mathrm{HP}(\mathbb{F}_{2k+1}, \mathbb{F}_{2j+1})$.

Theorem 32 *Suppose* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are posets with the strong HC-SCP property. Further suppose that the width of* $\mathbf{P}_2$ *is odd. Then* $\mathbf{P}_1 \times \mathbf{P}_2$ *has the HC-SCP property.*

Proof Let H_1 and H_2 be hamiltonian cycles that witness the fact that $\mathbf{P}_1$ and $\mathbf{P}_2$ have the strong HC-SCP property, respectively. By Proposition 19, we may assume that H_1 and H_2 have blocks of Type 2. Furthermore, by Theorem 25, we may assume the widths of $\mathbf{P}_1$ and $\mathbf{P}_2$ are odd. Therefore, H_1 and H_2 have an odd number of Type 2 blocks.

Suppose all blocks in H_1 and H_2 are Type 2. Then there are positive integers k and j such that $\mathbf{P}_1$ and $\mathbf{P}_2$ are isomorphic to $\mathbb{C}_{2k+1}$ and $\mathbb{C}_{2j+1}$, respectively. Lemma

27 allows us to find HP($\mathbb{F}_{2k+1}, \mathbb{F}_{2j+1}$), to which we add the edge from (a_1, a_1) to (b_{2k+1}, a_1) to find our desired hamiltonian cycle.

So, without loss of generality, we may assume that H_2 has a Type 1 block. Orient H_2 so that its first block, B_1, is Type 2 and U and its second block, B_2, is Type 1. Further, orient $\mathbf{P}_1$ so that it is in *standard position*. We then construct our desired hamiltonian cycle in the following way. Start by using the strategy developed in the proof of Theorem 25 (recall that the initial Type 1 blocks are traversed in the manner of an inside group). Continue to use this strategy until a Type 1 block $B_2 = \{C', C''\}$, with $\{|C'|, |C''|\} = \{2, 4\}$, is reached in H_2. According to Lemma 28, we must reach B_2 before encountering a second Type 2 block in H_2. Traverse $\mathbf{P}_1 \times B_2$ according to the method demonstrated in the proof of Lemma 29, which depends on the orientation of B_2. At this stage, we have visited all points in $\mathbf{P}_1 \times \mathbf{P}'$, where $\mathbf{P}'$ consists of all points in H_2 from B_1 to B_2. Let $\mathbf{P}'' = \mathbf{P}_2 - \mathbf{P}'$. Notice that $\mathbf{P}''$ has even width. Therefore, we can use the strategy developed in the proof of Theorem 25 on $\mathbf{P}_1 \times \mathbf{P}''$ to finish off our desired hamiltonian cycle. □

8 Conclusion

In this manuscript we have shown that, for all $n \geq 1$, $\mathcal{B}(n)$ has the strong HC-SCP property, and further that the strong HC-SCP property is weakly closed under cartesian products. Part of the motivation for this line of research was to make progress on the well-known Middle Two Levels conjecture, which states that, for every $n \geq 1$, the bipartite graph formed by the middle two levels of the subset lattice $\mathcal{B}(2n+1)$ is hamiltonian. While there has been much progress in the last 30 years (e.g. [4,7,13,14]), the conjecture remains open.

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References

1. Anderson, I.: Combinatorics of Finite Sets. Oxfordshire, Oxford; Clarendon Press, New York (1987)
2. de Bruijn, N.G., van Ebbenhorst Tengbergen, C., Kruyswijk, D.: On the set of divisors of a number. Nieuw Arch. Wisk **23**(2), 191–193 (1952)
3. Engel, K.: Sperner Theory. Cambridge University Press, New York (1997)
4. Felsner, S., Trotter, W.T.: Colorings of diagrams of interval orders and alpha-sequences of sets. Discret. Math. **144**, 23–31 (1995)
5. Greene, C., Kleitman, D.J.: Strong versions of Sperner's theorem. J. Combin. Theory Ser. A **20**(1), 80–88 (1976)
6. Griggs, J, Killian, C., Savage, C.: Venn diagrams and symmetric chain decompositions in the Boolean lattice. Electronic J. Combin. **11**(R2), 30 (2004)
7. Johnson, R.J.: Long cycles in the middle two layers of the discrete cube. J. Combin. Theory Ser. A **105**(2), 255–271 (2004)
8. Katona, G.O.H.: On a conjecture of Erdős and a stronger form of Sperner's theorem. Studia Sci. Math. Hungar. **1**, 59–63 (1966)
9. Kleitman, D.J.: On a lemma of Littlewood and Offord on the distribution of certain sums. Math. Z. **90**, 251–259 (1965)
10. Kleitman, D.J.: On a lemma of Littlewood and Offord on the distributions of linear combinations of vectors. Adv. Math. **5**, 155–157 (1970)

11. Littlewood, J.E., Offord, A.C.: On the number of real roots of a random algebraic equation (III). , (1943)
12. Savage, C.D.: A survey of combinatorial Gray codes. SIAM Rev. **39**(4), 605–629 (1997)
13. Savage, C.D., Winkler, P.: Monotone Gray codes and the middle levels problem. J. Combin. Theory Ser. A **70**(2), 230–248 (1995)
14. Savage, C.D., Shields, I.: A Hamilton path heuristic with applications to the middle two levels problem. Congr. Numer. **140**, 161–178 (1999)
15. Sperner, E.: Ein Satz über Untermengen einer endlichen Menge. Math. Zeit **27**, 544–548 (1928)
16. Streib, N.: Planar and Hamiltonian Cover Graphs. Ph.D. thesis, Georgia Institute of Technology (2012)
17. West, D.B.: Extremal Problems in partially ordered sets. In: Reidel, D.(ed.) Ordered Sets. (Proc. NATO ASI, Banff 1981, I. Rival), pp. 473–521 (1982)