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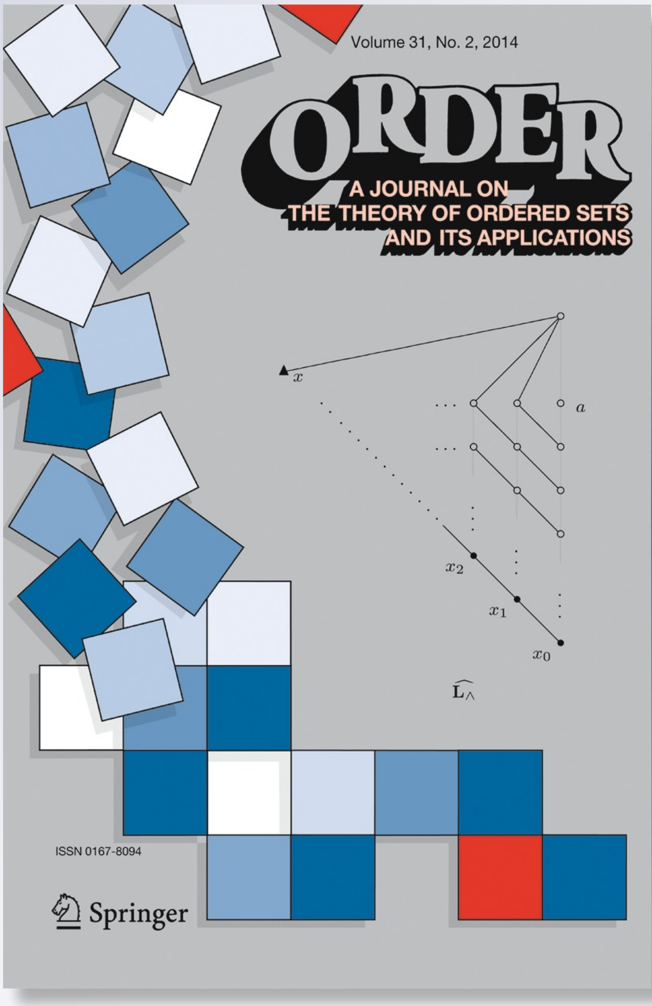

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Abstract We prove two theorems concerning incidence posets of graphs, cover graphs of posets and a related graph parameter. First, answering a question of Haxell, we show that the chromatic number of a graph is not bounded in terms of the dimension of its incidence poset, provided the dimension is at least four. Second, answering a question of Kříž and Nešetřil, we show that there are graphs with large girth and large chromatic number among the class of graphs having eye parameter at most two.

Keywords Chromatic number · Incidence poset · Dimension

Mathematics Subject Classifications (2010) 06A07 · 05C35

1 Introduction

The *chromatic number* of a graph $G = (V, E)$, denoted $\chi(G)$, is the least positive integer r for which there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_r$ of the vertex set V of G so that V_i is an independent set in G, for each $i = 1, 2, \ldots, r$. A family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of a poset P is a *realizer* of P if $P = \cap\mathcal{R}$, i.e., $x < y$ in P if and only if $x < y$ in L_i for each $i = 1, 2, \ldots, t$. The *dimension* of a poset P, denoted $\dim(P)$, is the minimum size of a realizer of P.

When $G = (V, E)$ is a graph, the *incidence poset* of G, denoted P_G, has $V \cup E$ as its ground set; vertices in V are minimal elements of P_G; edges in E are maximal

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elements of P_G; and a vertex x is less than an edge e in P_G when x is one of the two endpoints of e. Alternatively, considering the edges of a graph as two element subsets of the vertex set, the incidence poset P_G of a graph $G = (V, E)$ is just the set $V \cup E$ partially ordered by inclusion.

When x and y are points in a poset P with $x > y$ in P, we say x *covers* y in P when there is no point z with $x > z > y$ in P. Also, we let G_P denote the *cover* graph of P. The graph G_P has the elements of P as vertices with $\{x, y\}$ an edge in G_P when one of x and y covers the other in P. The *diagram* of P (also called a *Hasse* diagram or *order* diagram) is a drawing of the cover graph of G in the plane with x higher in the plane than y whenever x covers y in P.

1.1 Mathematical Preliminaries

We will find it convenient to work with the following alternative definition of dimension of an incidence poset, as proposed by Barrera-Cruz and Haxell [2].

Proposition 1.1 *Let $G = (V, E)$ be a graph and let P_G be its incidence poset. Then* $\dim(P_G)$ *is the least positive integer t for which there is a family $\{L_1, L_2, \ldots, L_t\}$ of linear orders on V so that the following two conditions are satisfied:*

(1) *If x, y and z are distinct vertices of G and $\{y, z\}$ is an edge in G, then there is some i with $1 \le i \le t$ for which both $x > y$ and $x > z$ in L_i.*
(2) *If x and y are distinct vertices of G, then there is some i with $1 \le i \le t$ so that $x > y$ in L_i.*

We note that some authors (see Hoşten and Morris [7], for example) have worked with a notion of *dimension* of a graph, which uses only the first of the two conditions listed in Proposition 1.1. This parameter is at most the dimension of the associated incidence poset. However, it can be at most one less. Furthermore, for connected graphs with no vertices of degree one, the two parameters agree.

In [8], Křížand Nešetřil defined a new parameter for graphs, which we call the *eye parameter*. Formally, the eye parameter of a graph G, denoted $\mathrm{eye}(G)$, is the least positive integer s for which there exists a family $\{L_1, L_2, \ldots, L_s\}$ of linear orders on the vertex set of G for which if x, y and z are three distinct vertices of G with $\{y, z\}$ an edge of G, then there is some i with $1 \le i \le s$ for which x is not between y and z in L_i. In this definition, it is allowed that x be *above* both y and z or *below* both y and z. For example, when G is a path, $\mathrm{eye}(G) = 1$. The following elementary proposition is stated for emphasis.

Proposition 1.2 *Let G be a graph and let P_G be the incidence poset of G. Then* $\mathrm{eye}(G) \le \dim(P_G) \le 2\,\mathrm{eye}(G)$.

2 Dimension and Chromatic Number

In this section, we state our two main theorems, including just enough background discussion to place them in context. Proofs are given in the two sections immediately following.

To understand the fundamental importance of incidence posets and dimension, we need only mention the following now classic theorem of Schnyder [10] (recently, a quite clever and very short proof has been provided by Barrera-Cruz and Haxell [2]).

Theorem 2.1 *Let G be a graph and let P_G be its incidence poset. Then G is planar if and only if* $\dim(P_G) \leq 3$.

In view of Schnyder's result, it is natural to ask the following two questions:

Question 1 Is the dimension of the incidence poset of a graph bounded in terms of the chromatic number of the graph?

Question 2 Is the chromatic number of a graph bounded in terms of the dimension of its incidence poset?

The first question was answered in the affirmative by Agnarsson, Felsner and Trotter in [1], where the following asymptotic formula is proved.

Theorem 2.2 *If G is a graph, P_G is the incidence poset of G and $\chi(G) = r$, then* $\dim(P_G) = O(\lg \lg r)$.

The inequality in Theorem 2.2 is best possible, up to the value of the multiplicative constant, since as noted in [1] (and by others), the dimension of the incidence poset P_r of the complete graph K_r is at least $\lg \lg r$, a statement which follows easily from repeated application of the Erdős/Szekeres theorem on monotonic sequences. On the other hand, Hoşten and Morris [7] showed that it is possible to determine the exact value of the dimension of P_r for surprisingly large values of r. Furthermore, a relatively tight asymptotic formula is known (see [1]):

$$\dim(P_r) = \lg \lg r + (1/2 + o(1)) \lg \lg \lg r.$$

It follows easily that if P is the incidence poset of a graph G with $\chi(G) = r$, then

$$\dim(P) \leq 2 \lg \lg r + (1 + o(1)) \lg \lg \lg r.$$

In view of Schnyder's theorem, we know the answer to the second question is yes, provided the dimension of the incidence poset is at most three. But, in this paper, we show that in general the answer to Question 2 is no, by proving the following theorem.

Theorem 2.3 *For every $r \geq 1$, there exists a graph G with $\chi(G) \geq r$ and* $\dim(P_G) \leq 4$.

Of course, the inequality $\dim(P_G) \leq 4$ in Theorem 2.3 will become tight once $r \geq 5$.

2.1 Cover Graphs

The cover graph of a poset is a triangle-free graph, and some thirty years ago, Rival asked whether there are cover graphs with large chromatic number. Bollobás noted in [4] that B. Descartes' classic proof [5] of the existence of triangle-free graphs with large chromatic number provided a positive answer. In fact, this construction shows that for each $r \geq 1$, there is a poset P of height r so that the chromatic number of the cover graph of P is r. In view of the dual form of Dilworth's theorem, this is the minimum value of height for which such a poset can possibly exist.

So with the fundamental question answered, several authors went on to prove somewhat stronger results. First, Bollobás [4] showed that there are lattices whose cover graphs have arbitrarily large chromatic number. Second, the construction of Nešetřil and Rödl [9] for graphs (and hypergraphs) with large girth and large chromatic number also implies that for each pair (g, r) of positive integers, there is a poset P of height r whose cover graph G_P has girth at least g and chromatic number r.

In another direction, as is well known, the shift graph S_n consisting of all 2-element subsets of $\{1, 2, \ldots, n\}$ with $\{i, j\}$ adjacent to $\{j, k\}$ when $1 \leq i < j < k \leq n$ is a cover graph of an interval order and has chromatic number $\lceil \lg n \rceil$. In this case, the height of the associated interval order is $n - 1$, and it was shown by Felsner and Trotter in [6] that the height of an interval order must be exponentially large in terms of the chromatic number of its cover graph. In fact, they conjecture that if P is an interval order whose height is at most $2^{r-1} + \lfloor (r-1)/2 \rfloor$, then the chromatic number of the cover graph of P is at most r. If true, this statement is best possible. Efforts to resolve this conjecture have led to interesting problems on hamiltonian paths in the subset lattice (see [3] and [11] for quite recent work on this theme).

Finally, we mention the work of Kříž and Nešetřil [8] answering a question posed by Nešetřil and Trotter by proving the following theorem, as this work is central to the results presented here.

Theorem 2.4 *For every $r \geq 1$, there exists a poset P with $\dim(P) \leq 2$ so that the chromatic number of the cover graph of P is r.*

If G is the cover graph of poset P with $\dim(P) \leq 2$, then $\text{eye}(G) \leq 2$, so as Kříž and Nešetřil noted, we have the following immediate corollary.

Corollary 2.5 *For every $r \geq 1$, there is a graph G with $\text{eye}(G) \leq 2$ and $\chi(G) = r$.*

The graphs constructed by Kříž and Nešetřil in the proof of Theorem 2.4 and Corollary 2.5 have girth four. However, they were able to prove the following extension.

Theorem 2.6 *For every pair (g, r) of positive integers, there is a graph G with $\text{eye}(G) \leq 3$, $\text{girth}(G) \geq g$ and $\chi(G) = r$.*

They asked whether this result remains true if we require $\text{eye}(G) \leq 2$. Our second main result will be to answer this question in the affirmative by proving the following theorem, which is in fact a slightly stronger result.

Theorem 2.7 *For every pair (g, r) of positive integers, there is a poset $P = P(g, r)$ with cover graph $G = G(g, r)$ so that the height of P is r, while $\text{girth}(G) \geq g$ and $\chi(G) = r$. Furthermore, there are two linear extensions L_1 and L_2 of P witnessing that $\text{eye}(G) \leq 2$.*

The reader should note that we do not claim that the poset $P(g, r)$ in Theorem 2.7 is 2-dimensional. In fact, the dimension of $P(g, r)$ grows rapidly with r, even with g fixed. We will return to this issue in the last section of this paper.

3 Proof of the First Main Theorem

We first explain why Theorem 2.3 follows as a relatively straightforward corollary to Theorem 2.4, starting with a lemma which we believe is of independent interest.[1]

Lemma 3.1 *Let P be a poset, let G be the cover graph of P and let Q be the incidence poset of P. Then $\dim(Q) \leq 2\dim(P)$.*

Proof Let $t = \dim(P)$ and let $\mathcal{R} = \{L_i : 1 \leq i \leq t\}$ be a realizer of P. Then for each $i = 1, 2, \ldots, t$, let L_i^d be the dual of L_i, i.e., $x > y$ in L_i^d if and only if $x < y$ in L_i. We claim that the family $\mathcal{R}^* = \mathcal{R} \cup \{L_i^d : 1 \leq i \leq t\}$ witnesses that $\dim(Q) \leq 2t$.

To see this, note that the second condition of Proposition 1.1 holds since L_1 and L_1^d are in the family. Now let x, y and z be distinct vertices with $\{y, z\}$ an edge in G. Without loss of generality, we take $y < z$ in P. If $x \not< z$ in P, then there is some i with $1 \leq i \leq t$ so that $x > z$ in L_i. This implies $x > z > y$ in L_i. So we may assume that $x < z$ in P. Since $\{y, z\}$ is an edge of the cover graph, we cannot have $y < x$ in P. It follows that there is some j with $1 \leq j \leq t$ so that $x < y$ in L_j. This implies that $x < y < z$ in L_j and $x > y > z$ in L_j^d. This completes the proof of the lemma. □

We now show how Theorem 2.3 follows as an easy corollary to Theorem 2.4. Let $r \geq 1$ and let P be the poset from Theorem 2.4. Then let $G = G_P$ be the cover graph of the poset P, noting that $\chi(G) \geq r$. Since $\dim(P) \leq 2$, from Lemma 3.1, we know that the dimension of the incidence poset of G is at most four.

4 Proof of the Second Main Theorem

We fix an integer $g \geq 4$ and then argue by induction on r. The basic idea behind the proof will be to make a minor adjustment to the construction used by Nešetřil and Rödl in [9]. The cases $r = 1$ and $r = 2$ are trivial. To handle the case $r = 3$, we let n be an odd integer with $n \geq g$. Then we take $G = G(g, 3)$ as an odd cycle with vertex set $\{a_1, a_2, \ldots, a_n\}$, with $\{a_i, a_{i+1}\}$ an edge for each $i = 1, 2, \ldots, n - 1$. Also, $\{a_n, a_1\}$ is

[1]We thank an anonymous referee for pointing out that our original manuscript included this lemma implicitly.

an edge of G. Then we take $P = P(g, 3)$ as a poset whose cover graph is G by setting the following covering relations in P:

$$a_1 < a_2 < a_3 > a_4 < a_5 > a_6 < a_7 > a_8 < a_9 > \cdots > a_{n-1} < a_n > a_1.$$

We then take

$$L_1 = a_1 < a_2 < a_4 < a_3 < a_6 < a_5 < a_8 < a_7 < \cdots < a_{n-1} < a_{n-2} < a_n$$

and

$$L_2 = a_{n-1} < a_1 < a_n < a_{n-3} < a_{n-2} < \cdots < a_6 < a_7 < a_4 < a_5 < a_2 < a_3.$$

It is easy to see that L_1 and L_2 are linear extensions of P. Furthermore, the two endpoints of an edge in G occur consecutively in either L_1 or L_2, except for the edge $\{a_{n-1}, a_n\}$. However, only a_{n-2} is between a_{n-1} and a_n in L_1. Also, only a_1 is between a_{n-1} and a_n in L_2. It follows that L_1 and L_2 witness that eye$(G) \leq 2$.

Now suppose that for some $r \geq 3$, we have constructed a poset $P = P(g, r)$ with cover graph $G = G(g, r)$ so that the height of P is r, while girth$(G) \geq g$ and $\chi(G) = r$. Suppose further that L_1 and L_2 are linear extensions of P witnessing that eye$(G) \leq 2$.

We now explain how to construct a poset $Q = P(g, r+1)$ with cover graph $H = G(g, r+1)$ so that the height of Q is $r+1$, while girth$(H) \geq g$ and $\chi(H) = r+1$. We will also construct linear extensions M_1 and M_2 of Q witnessing that eye$(H) \leq 2$. As the reader will sense, there is considerable flexibility in how these steps are taken, and our approach is an effort to make the exposition as clear as possible.

Let A denote the vertex set of G and let $n = |A|$. Using the results of Nešetřil and Rödl as developed in [9], we know there exists a hypergraph $\mathcal{H}$ satisfying the following conditions: $\mathcal{H}$ is a simple n-uniform hypergraph; the girth of $\mathcal{H}$ is at least g; and the chromatic number of $\mathcal{H}$ is $r+1$. Let B and $\mathcal{E}$ denote, respectively, the vertex set and the edge set of $\mathcal{H}$. In the discussion to follow, we consider each edge $E \in \mathcal{E}$ as an n-element subset of B.

The poset Q is assembled as follows. Set $Z = \mathcal{E} \times A$. The ground set of Q will be $B \cup Z$ with all elements of B maximal in Q. For each edge E in $\mathcal{E}$, the elements of $\{E\} \times A$ determine a subposet of Q which we will denote $P(E)$. When a and a' are distinct elements of A, we will set $(E, a) < (E, a')$ in Q if and only if $a < a'$ in P. Accordingly, for each $E \in \mathcal{E}$, the subposet $P(E)$ is isomorphic to P. Also, when $E, E' \in \mathcal{E}$ and $E \neq E'$, we make all elements of $P(E)$ incomparable with all elements of $P(E')$.

We pause to point out that regardless of how the comparabilities between B and Z are defined in Q, for each edge $E \in \mathcal{E}$, the covering edges of $P(E)$ are covering edges in Q and these edges form a copy of G.

We now describe these comparabilities between B and Z. This will be done by prescribing when an element $b \in B$ covers an element $(E, a) \in Z$. We begin by choosing an arbitrary linear order $L(B)$ on B. Also, let $\{a_1, a_2, \ldots, a_n\}$ be a labelling of A so that L_1 is the subscript order, i.e., $a_i < a_j$ in L_1 if and only if $i < j$. Next, we fix an edge $E \in \mathcal{E}$ and describe the cover relations between B and $P(E)$. This process will be repeated for each edge $E \in \mathcal{E}$ and when this step has been completed, the poset Q is fully determined. First, when $b \in B - E$, we make b incomparable to all elements of $P(E)$ in Q. Second, let $\{b_1, b_2, \ldots, b_n\}$ be the labelling of the elements of E so that $b_i < b_j$ in $L(B)$ if and only if $i < j$. Then for each $i = 1, 2, \ldots, n$, we make

b_i cover (E, a_i) in Q. It follows that if $(E, a) \in Z$, then there is a unique element $b \in B$ so that b covers (E, a) in Q.

Now that $Q = P(G, r+1)$ has been defined, we take $H = G(g, r+1)$ as the cover graph of Q, and we pause to show that the height of Q is $r+1$, while girth$(H) \geq g$ and $\chi(H) = r+1$. First, we note that the height of Q is at most $r+1$, since we have added B as a set of maximal elements to a family of pairwise disjoint and incomparable copies of P. On the other hand, once we have shown that $\chi(H) = r+1$, we will have also shown that the height of H must be $r+1$, using the dual form of Dilworth's theorem.

Second, we note that $\chi(H) \geq r$, since H contains copies of G. On the other hand, it is trivial that we may color all elements of Z with r colors and use one new color on the independent set B, so that $\chi(H) \leq r+1$. Now suppose that $\chi(H) = r$, and let ϕ be a proper coloring of H using r colors. Then since the chromatic number of $\mathcal{H}$ is $r+1$, there is some edge E of $\mathcal{H}$ on which ϕ is constant. This implies that ϕ colors the cover graph of $P(E)$ with only $r-1$ colors, which is impossible. The contradiction shows that $\chi(H) = r+1$, as desired.

Third, we show that the girth of H is at least g. Consider a cycle C in H. If there is an edge E of $\mathcal{H}$ so that C is contained entirely within the cover graph of $P(E)$, then it has size at least g. So we may assume that C involves vertices from copies of P associated with two or more edges in $\mathcal{E}$. Now the fact that the covering edges between Y and each $P(E)$ are formed using a bijection means that once the cycle enters some $P(E)$, it must pass through at least two vertices before leaving. So the girth requirement is satisfied (generously) be the fact that the girth of $\mathcal{H}$ is at least g. In this detail, we point out that we are using essentially the same idea as in [9].

Now we turn our attention to the eye parameter. To complete the proof, we must construct two linear extensions M_1 and M_2 of Q witnessing that eye$(H) \leq 2$. As we remarked previously, there is considerable flexibility in how this is done.

For each $b \in Y$, let $N(b)$ denote the set of all elements (E, a) from Z such that b covers (E, a) in Q, i.e., $N(b)$ is just the neighborhood of b in the cover graph H. Note that $N(b)$ is an antichain in the poset Q.

Let $L(\mathcal{E})$ be an arbitrary linear order on $\mathcal{E}$. We define linear extensions M_1 and M_2 by the following rules (starting with the rules for M_2):

(1) The restriction of M_2 to B is an arbitrary linear order. In M_2 all elements of Z are below all elements of Y. Furthermore, if (E, a) and (E', a') are distinct elements of Z, then $(E, a) < (E', a')$ in M_2 if and only if either $E < E'$ in $L(\mathcal{E})$ or $E = E'$ and $a < a'$ in L_2.
(2) The restriction of M_1 to B is the linear order $L(B)$. In M_1, for each $b \in B$, all elements of $N(b)$ will be placed in the gap immediately under b and above all other elements (if any) of B which are under b in $L(B)$. The restriction of M_1 to $N(b)$ will be the dual of the restriction of M_2 to $N(b)$.

We pause to show that M_1 and M_2 are linear extensions of Q, and we remark that it is enough to show that they both respect the covering relations in Q. First, we note that for each $i = 1, 2$, and for each $E \in \mathcal{E}$, if a and a' are distinct elements of A, then $(E, a) < (E, a')$ in M_i if and only if $a < a'$ in L_i. On the other hand, if $b \in B$, $(E, a) \in Z$ and b covers (E, a) in Q, then $(E, a) \in N(b)$ so it is placed below b in M_1. Finally, we note that all elements of Z are below all elements of B in M_2. We conclude that M_1 and M_2 are linear extensions of Q, as desired.

Finally, we explain why M_1 and M_2 witness that $\text{eye}(H) \le 2$. Consider how an edge might possibly trap a vertex in both M_1 and M_2. If the edge is an edge in the cover graph of some $P(E)$, then the linear extension M_2 forces the vertex to also belong to $P(E)$. But the restriction of M_1 and M_2 to $P(E)$ are just like L_1 and L_2 for G, so this situation cannot lead to a problem.

Similarly, if the edge joins some $b \in Y$ to a vertex (E, a) in $N(b)$, then the only potential problem is a vertex $(E', a') \in N(b)$ with $(E, a) < (E', a') < y$ in M_1. However, the rules for M_1 and M_2 imply that $(E', a') < (E, a) < b$ in M_2. This completes the proof of Theorem 2.7.

5 Conjectures and Questions

We have made some effort, without success, to construct a poset P with cover graph G so that the dimension of P is small; the girth of G is large; and the chromatic number of G is large. Accordingly, we believe it reasonable to make the following conjecture.

Conjecture 5.1 For every pair (g, d) of integers, with $g \ge 5$ and $d \ge 1$, there is an integer $r = r(g, d)$ so that if G is the cover graph of a poset P, $\dim(P) \le d$ and $\text{girth}(G) \ge g$, then $\chi(G) \le r$.

In another direction, we return to Proposition 1.2 and make the following conjectures.

Conjecture 5.2 For every $t \ge 1$, there is a graph G so that if P_G is the incidence poset of G, then $\text{eye}(G) = \dim(P_G)$.

Conjecture 5.3 For every $t \ge 1$, there is a graph H so that if P_H is the incidence poset of H, then $\dim(P_H) = 2\,\text{eye}(H)$.

A similar analysis of Lemma 3.1 leads to the following conjectures.

Conjecture 5.4 For every $t \ge 1$, there exists a poset P so that if G is the cover graph of P and Q is the incidence poset of G, then $\dim(Q) = 2\dim(P)$.

Clearly, Conjecture 5.4 holds when $t \le 2$, but we have not been able to settle the issue for larger values of t. However, our preliminary thoughts on this conjecture suggest a more extensive line of research. For a poset P, we call a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P an *upper-cover realizer of* P provided that whenever (z, x, y) is an ordered triple of distinct points in P with z covering both x and y, there is some i with $x > y$ in L_i. The *upper cover dimension* of P, denoted $\dim_{uc}(P)$, would then be the minimum size of an upper-cover realizer of P.

Lower-cover realizers and the lower cover dimension of P, denoted $\dim_{lc}(P)$, would then be defined dually. Clearly, $\dim_{uc}(P) \le \dim(P)$ and $\dim_{lc}(P) \le \dim(P)$.

An attractive feature of these new parameters is that they are monotonic on subdiagrams of the order diagram of P, i.e., if we consider the diagram D of P as an acyclic orientaton of the cover graph G, and D' is a subdiagram of D, then D'

determines a poset P' which is a suborder of P. On the one hand, P' is not necessarily a subposet of P and it is clear that $\dim(P')$ may be the same as $\dim(P)$ or arbitrarily smaller or larger. On the other hand, $\dim_{uc}(P') \leq \dim_{uc}(P)$ and $\dim_{lc}(P') \leq \dim_{lc}(P)$.

We make the following conjecture, which is easily seen to be stronger than Conjecture 5.4.

Conjecture 5.5 For every pair (d, r) of positive integers, there is a poset P with $\dim(P) = d$ so that if D is the order diagram of P, E is the edge set of D and $\phi : E \to \{1, 2, \ldots, r\}$ is an r coloring of the edges of D, then there is some $\alpha \in \{1, 2, \ldots, r\}$ so that if we take D' as the subdiagram of D with edge set $\{e \in E : \phi(e) = \alpha\}$ and set P' as the suborder of P determined by D', then $\dim_{uc}(D') = \dim_{lc}(D') = \dim(P)$.

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Note added in proof The authors have just learned that Theorem 2.3 was proved previously by Patrice Ossona de Mendez and Pierre Rosenstiehl in their paper titled Homomorphism and Dimension, appearing in Combinatorics, Probability and Computing, 14 (2005), 861–872.

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