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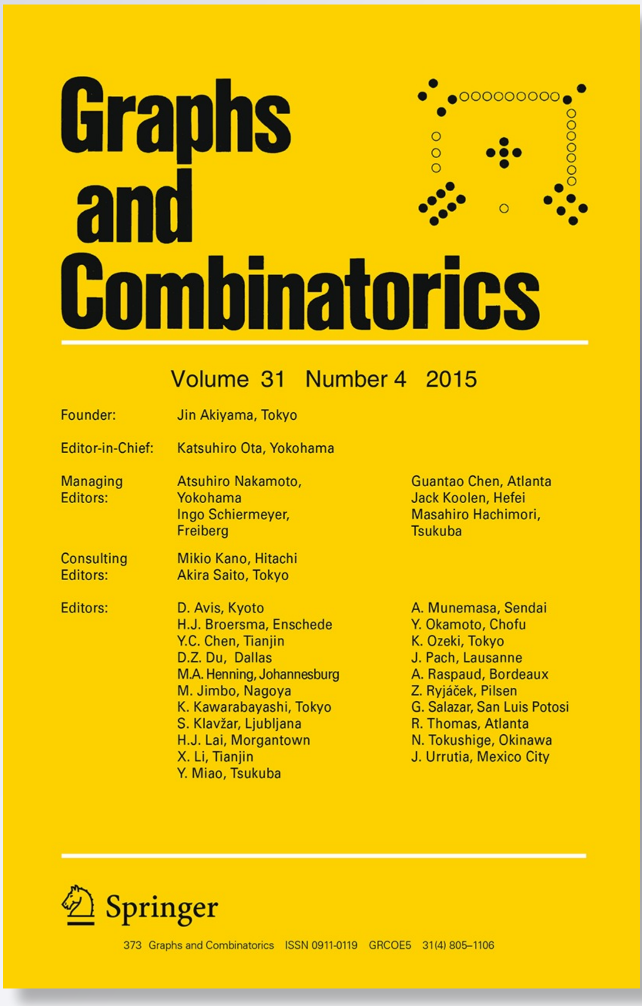

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ORIGINAL PAPER

The Dimension of Posets with Planar Cover Graphs

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Abstract Kelly showed that there exist planar posets of arbitrarily large dimension, and Streib and Trotter showed that the dimension of a poset with a planar cover graph is bounded in terms of its height. Here we continue the study of conditions that bound the dimension of posets with planar cover graphs. We show that if P is a poset with a planar comparability graph, then the dimension of P is at most four. We also show that if P has an outerplanar cover graph, then the dimension of P is at most four. Finally, if P has an outerplanar cover graph and the height of P is two, then the dimension of P is at most three. These three inequalities are all best possible.

Keywords Cover graph · Comparability graph · Outerplanar graph · Dimension

Mathematics Subject Classification (2000) 06A07 · 05C35

1 Introduction

We are concerned here with combinatorial problems for partially ordered sets. We assume some familiarity with concepts and results in this area, including cover graphs, comparability graphs and dimension. We also assume that the reader is familiar with the notions of planarity and outerplanarity for graphs, as well as the concept of planarity

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for posets, i.e. posets which have planar order diagrams. For readers who are new to the subject, we suggest consulting the second author's monograph [14] and survey article [15] for background information.

1.1 Posets and Planarity

We recall three key results relating dimension and planarity. The first result is essentially due to Baker et al. [2], although the stronger form we present here was developed in the interim and may be considered as part of the folklore of the subject.

Theorem 1.1 *Let P be a poset with a zero and a one. Then P is planar if and only if P is a 2-dimensional lattice.*

The second result is due to Trotter and Moore [16].

Theorem 1.2 *Let P be a poset with a zero (or a one). If P is planar, then* $\dim(P) \leq 3$.

We show in Fig. 1 some 3-dimensional posets which remain planar when a zero is attached. Each of these posets is in fact 3-*irreducible*, i.e., if any point is removed, the dimension drops to two.

For $n \geq 2$, the *standard example* S_n is the height 2 poset having n minimal elements $a_1, a_2, \ldots, a_n$ and n maximal elements $b_1, b_2, \ldots, b_n$ with $a_i < b_j$ if and only if $i \neq j$. The dimension of S_n is n; also, when $n \geq 3$, S_n is n-irreducible.

It is easy to see that the standard example S_4 is planar, so there exists a 4-dimensional planar poset. On the other hand, for each $n \geq 5$, the standard example S_n has a non-planar cover graph. However, Kelly [9] showed for each $n \geq 2$, S_n is a subposet of a planar poset. We illustrate Kelly's construction in Fig. 2 where we show S_6 as a subposet of a planar poset. Clearly, the construction can be modified for all $n \geq 2$.

1.2 Adjacency Posets and Height

When $G = (V, E)$ is a finite graph, the *adjacency poset* of G is the height 2 poset P_G with (1) minimal elements $\{x' : x \in V\}$; (2) maximal elements $\{x'' : x \in V\}$; and (3) order relation $x' < y''$ in P_G when $xy \in E$. Felsner et al. [8] investigated the adjacency posets of planar and outerplanar graphs, establishing the following inequalities.

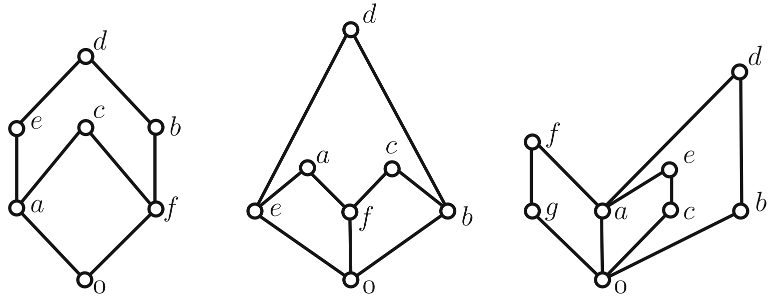

Fig. 1 Some 3-dimensional posets

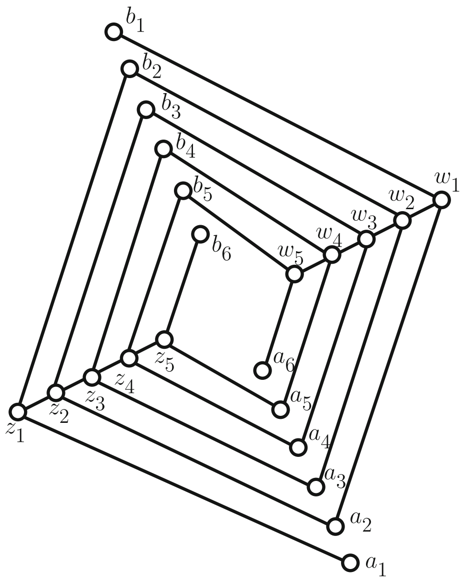

Fig. 2 Planar posets with large dimension

Theorem 1.3 *If P_G is the adjacency poset of a planar graph G, then* $\dim(P_G) \leq 8$.

Theorem 1.4 *If P_G is the adjacency poset of an outerplanar graph G, then* $\dim(P_G) \leq 5$.

It is not known whether Theorems 1.3 and 1.4 are best possible. However, it is shown in [8] that there exists a planar graph whose adjacency poset has dimension 5, and there is an outerplanar graph whose adjacency poset has dimension 4.

When P is a a connected poset of height 2, the comparability graph G_P of P is the same as the cover graph of P, a bipartite graph with the maximal elements on one side and the minimal elements on the other. Furthermore, the adjacency poset P_{G_P} of G_P is then the disjoint sum $P + P^d$ of P and its dual. Moreover, when P is not a chain, P, P^d and $P + P^d$ all have the same dimension.

Conversely, a connected non-trivial bipartite graph G is the cover graph—and the comparability graph—of a poset P of height 2. Furthermore, if G is planar, then the adjacency poset P_G of G is again the disjoint sum $P + P^d$. Also, as detailed in [8], both P and P^d are isomorphic to the vertex-face poset of a planar multigraph. Applying the results of Brightwell and Trotter [3,4], the following inequality was proved in [8].

Theorem 1.5 *If P is a poset with a planar cover graph and the height of P is at most* 2*, then the dimension of P is at most* 4.

The standard example S_4 shows that the inequality of Theorem 1.5 is best possible. Moreover, the fact that the posets in Kelly's construction have large height led the authors of [8] to conjecture that the dimension of a poset with a planar cover graph is bounded in terms of its height. The conjecture has recently been settled in the affirmative by Streib and Trotter [12].

Theorem 1.6 *For every $h \geq 1$, there exists a constant c_h so that if P is a poset with a planar cover graph and the height of P is at most h, then the dimension of P is at most c_h.*

The value $c_1 = 2$ is trivial, as a poset of height 1 is an antichain. On the other hand, the results of [8] show that $c_2 = 4$. However, it should be pointed out that this is a very non-trivial result, as in fact, there is no entirely elementary argument which shows that the dimension of height 2 posets with a planar cover graph is bounded. Moreover, the arguments used by Streib and Trotter to prove Theorem 1.6 for $h \geq 3$ use Ramsey theoretic techniques, so the bounds are likely to be far from best possible.

1.3 Dimension and Boxicity

Adiga et al. [1] develop connections between the boxicity of graphs and the dimension of posets. In particular, they show that if G_P is the comparability graph of a poset P, then $\dim(P) \leq 2\,\mathrm{box}(G_P)$. Thomassen [13] showed that the boxicity of a planar graph is at most three. As noted in [1], it follows that the dimension of a poset P with a planar comparability graph is at most 6. A poset with a planar comparability graph has height at most 4, so we know that its dimension is at most the constant c_4 from Theorem 1.6. However, as we have already remarked, the constant c_4 guaranteed by the proof is very large and almost certainly far from being best possible.

Adiga et al. [1] also noted that Scheinerman [11] had proved that the boxicity of an outerplanar graph is at most two. As a consequence, their techniques imply that if P is a poset with an outerplanar comparability graph, then the dimension of P is at most four.

1.4 Our Results

The principal results of this paper are a continuation of the study of conditions that bound the dimension of posets with planar cover graphs. Our first theorem concerns posets with planar comparability graphs. It tightens the inequality of Adiga et al. [1] and strengthens Theorem 1.5.

Theorem 1.7 *If P is a poset with a planar comparability graph, then* $\dim(P) \leq 4$.

The standard example S_4 shows that the inequality of Theorem 1.7 is best possible.

Our next two results concern posets with outerplanar cover graphs, a class of posets that can have arbitrarily large height. Note that the result of [1] for posets with outerplanar comparability graphs is limited to posets of height at most 4.

Theorem 1.8 *If P is a poset with an outerplanar cover graph, then* $\dim(P) \leq 4$.

In Sect. 4, we will show that the inequality in Theorem 1.8 is best possible. The poset constructed there has height 3. For posets of height 2, we can say more.

Theorem 1.9 *If P is a poset with an outerplanar cover graph and the height of P is* 2, *then* $\dim(P) \leq 3$.

All three posets shown in Fig. 1 have outerplanar cover graphs and the last two have height 2. This shows that the inequality of Theorem 1.9 is best possible.

2 The Proofs—Part 1

Our proof for Theorem 1.7 will depend on work done by Brightwell and Trotter [3,4] where the following theorem (stated here in comprehensive form) is proved.

Theorem 2.1 *Let $\mathcal{D}$ be a drawing without edge crossings of a planar multigraph G, and let Q_G denote the vertex-edge-face poset determined by $\mathcal{D}$. Then* $\dim(Q_G) \leq 4$. *Furthermore, if G is a simple graph* (*no loops or multiple edges*) *and G is* 3*-connected, then the subposet P_G determined by the vertices and faces is* 4*-irreducible.*

Readers who would like to explore the proofs of the statements in Theorem 2.1 are encouraged to investigate the two papers [6,7] where polished arguments in a more comprehensive setting are given.

We will also require a concept introduced by Kimble [10], the notion of a *split*. When S is a non-empty subset of the ground set of a poset P, the *split* of S is the poset Q whose ground set is $(P - S) \cup \{x' : x \in S\} \cup \{x'' : x \in S\}$. The order relation on Q is defined as follows:

1. The restriction of Q to $P - S$ is the same as the subposet $P - S$ of P.
2. The elements of $\{x' : x \in S\}$ are minimal in Q.
3. The elements of $\{x'' : x \in S\}$ are maximal in Q.
4. If $x, y \in S$, then $x' < y''$ in Q when $x \leq y$ in P.
5. If $x \in S$ and $u \in P - S$, then $x' < u$ in Q when $x < u$ in P.
6. If $x \in S$ and $u \in P - S$, then $u < x''$ in Q when $u < x$ in P.

Kimble [10] proved the following inequalities for splits.

Theorem 2.2 *Let S be a non-empty subset of the ground set of a poset P and let Q be the split of S. Then* $\dim(P) \leq \dim(Q) \leq 1 + \dim(P)$.

In the argument to follow, we will apply the special case of Theorem 2.2 when S consists of a single point.

Now on to the proof. Clearly, we may restrict our attention to posets without isolated points. For such a poset P, let $P = X \cup S \cup Y$ be the partition of the ground set of P defined by setting X to be the set of minimal elements and Y the set of maximal elements. The remaining elements, neither minimal nor maximal belong to S. We then assume that Theorem 1.7 is false and that there exists a poset P with a planar comparability graph such that $\dim(P) \geq 5$. Of all such posets, we may assume that P has been chosen so that the quantity $q(P) = 3|S| + |X| + |Y|$ is as small as possible. Clearly, P is a connected poset. In fact, P is 5-irreducible—although that detail will not be essential to the argument.

Throughout the remainder of the argument, we fix a plane drawing $\mathcal{D}$ (with no edge crossings) of the comparability graph of P. Without loss of generality, we may assume the edges in this drawing are straight line segments.

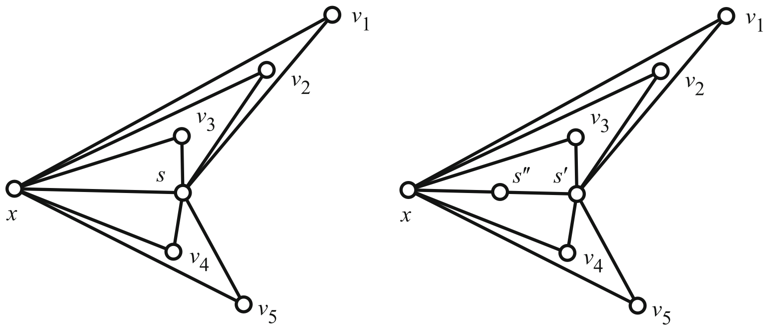

Fig. 3 Splitting a point

We note that if $S = \emptyset$, then P has height 2, and it would follow from Theorem 1.5 that $\dim(P) \leq 4$. Accordingly, we know that $S \neq \emptyset$. For each $s \in S$, let $D(s) = \{u \in P : u < s\}$ and $U(s) = \{v \in P : s < v\}$. Both $D(s)$ and $U(s)$ are non-empty. If $\min\{|D(s)|, |U(s)|\} \geq 2$ and $\max\{|D(s)|, |U(s)|\} \geq 3$, then the comparability graph of P would contain the complete bipartite graph $K_{3,3}$ and would be non-planar.

Now suppose that $s \in S$ and $|D(s)| = 1$. Let $x \in X$ be the unique element of $D(s)$. If Q is the split of $\{s\}$, then we know that $\dim(Q) \geq \dim(P) \geq 5$. Also, the quantity $q(Q)$ is less than $q(P)$. Furthermore, the poset Q has a planar comparability graph. To see this, focus on the edge $e = xs$ in the drawing $\mathcal{D}$. Modify $\mathcal{D}$ by inserting a new point s'' in the middle of e and relabel s as s'. The resulting drawing witnesses that the comparability graph of Q is planar. We illustrate this situation in Fig. 3, where we have $|D(s)| = 1$ and $|U(s)| = 5$.

The contradiction shows that $|D(s)| \geq 2$ for every $s \in S$. A dual argument shows that $|U(s)| \geq 2$ for every $s \in S$. Thus P has height 3 and S is an antichain. Furthermore, for every $s \in S$, $D(s)$ is a 2-element subset of X and $U(s)$ is a 2-element subset of Y.

Now let $s \in S$ with $D(s) = \{x_1, x_2\} \subseteq X$ and $U(s) = \{y_1, y_2\} \subseteq Y$. The comparability graph G_P of P contains the following eight edges:

$$x_1y_1, x_2y_1, x_1y_2, x_2y_2, sx_1, sx_2, sy_1, sy_2.$$

In the drawing $\mathcal{D}$, consider the four edges incident with s. As illustrated in Fig. 4, either these edges occur in blocks as shown on the left or they alternate as shown on the right. However, if they occur in blocks, then we may add to G_P the edges x_1x_2 and y_1y_2 and preserve planarity. This would be a contradiction since the five vertices in $\{s, x_1, x_2, y_1, y_2\}$ would then form the complete graph K_5. It follows that the four edges always alternate as shown on the right. To obtain the final contradiction, we will now prove the following claim.

Claim *The poset P is isomorphic to a subposet of the vertex-edge-face poset of the planar map determined by a drawing of a planar multigraph.*

Proof The argument for this claim follows along similar lines as the technique used by by Felsner, Li and Trotter in the proof of Theorem 1.5. The overall scheme is that we will modify the plane drawing $\mathcal{D}$ of the comparability graph of P into a drawing

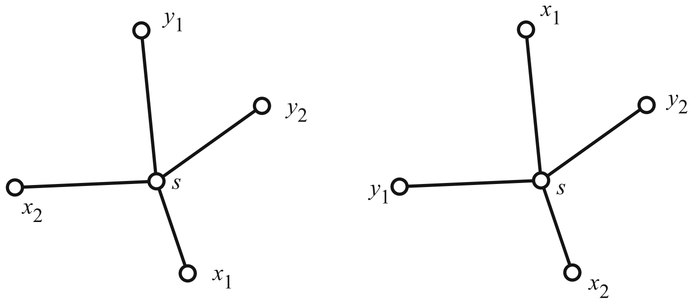

Fig. 4 Four neighbors—two cases

$\hat{\mathcal{D}}$, without edge crossings, of a planar multigraph. The vertex set of $\hat{\mathcal{D}}$ is the set X of minimal elements. Each element $s \in S$ will correspond to an edge e_s in $\hat{\mathcal{D}}$, although there will be edges in $\hat{\mathcal{D}}$ that do not correspond to elements of S. Similarly, with each $y \in Y$, we will associate a face F_y in $\hat{\mathcal{D}}$. These correspondences will satisfy the following conditions:

1. If $x \in X$, $s \in S$, then $x < s$ in P if and only if vertex x is an end point of edge e_s in $\hat{\mathcal{D}}$.
2. If $x \in X$, $y \in Y$, then $x < y$ in P if and only if vertex x is a vertex on the boundary of face F_y in $\hat{\mathcal{D}}$.
3. If $s \in S$, $y \in Y$, then $s < y$ in P if and only if edge e_s is part of the boundary of face F_y in $\hat{\mathcal{D}}$.

In this multigraph, loops and multiple edges are allowed.

The construction is described in terms of two parameters, ϵ and δ. The value of ϵ will be specified as the minimum distance in the drawing $\mathcal{D}$ between distinct vertices in the comparability graph G_P. At the end of the argument, it will be clear that all the desired properties hold, provided δ is sufficiently small.

First, the elements of X will be the vertices positioned in $\hat{\mathcal{D}}$ exactly the same as in the original drawing $\mathcal{D}$.

Second, each element $y \in Y$ will be transformed into a face F_y in $\hat{\mathcal{D}}$. To start this transformation, we consider a circle C_y of radius $\epsilon/3$ whose center is the point in the plane where the vertex y is located in $\mathcal{D}$. Note that C_y and $C_{y'}$ are disjoint when y and y' are distinct elements of Y. Also note that y is the only vertex of G_P inside or on the boundary of C_y.

In the revised drawing $\hat{\mathcal{D}}$, the interior of C_y will be contained in the interior of F_y and a portion of the boundary of C_y will belong to the boundary of F_y. However, F_y will contain some “protrusions” from C_y to account for elements of $X \cup S$ which are less than y in P.

Now fix an element $y \in Y$. We provide additional detail on how the face F_y is constructed. As a starter, keep in mind the fact that the vertex y will not be present in $\hat{\mathcal{D}}$ and only portions of the line segments incident with the vertex y in $\mathcal{D}$ will remain in $\hat{\mathcal{D}}$. These segments will be part of the boundary of F_y.

Let $s \in S$ with $s < y$ in P. Then let $D(s) = \{x_1, x_2\}$. From our previous remarks, we know that there is an element $y' \in Y$ so that $U(s) = \{y, y'\}$. We assume that in

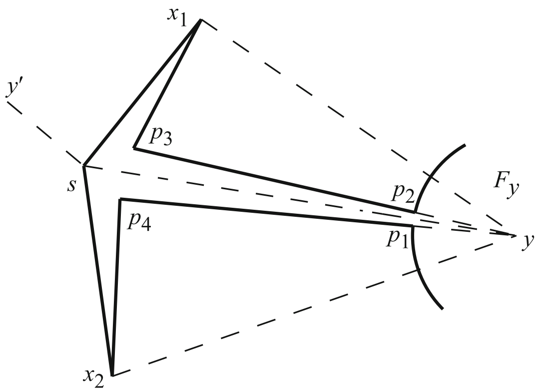

Fig. 5 Expanding the face—case 1

clockwise order, the four edges incident with s in $\mathcal{D}$ occur in clockwise order as sx_1, sy, sx_2, sy'. If this order is reversed, the discussion to follow must be modified in an obvious manner.

In the drawing $\hat{\mathcal{D}}$, we consider a ray r_1 emanating from y at a clockwise angle δ before the line ys and intersecting the circle C_y at a point p_1. We also consider a ray r_2 emanating from y at a clockwise angle δ after the ray ys and intersecting C_y at p_2. Then consider a ray r_3 emanating from x_1 at a clockwise angle δ before x_1s and intersecting ray r_2 at p_3. Similarly, consider a ray r_4 emanating from x_2 at a clockwise angle δ after x_2s and intersecting ray r_1 at a point p_4.

Then we delete the arc of C_y between p_1 and p_2 and add the line segments p_1p_4, x_2p_4, x_1p_3 and p_3p_2. Finally we delete the vertex s and the line segment sy, but leave the line segments sx_1 and sx_2 intact. Together, these last two segments form the edge e_s in $\hat{\mathcal{D}}$ with x_1 and x_2 as its end points. Also the edge e_s is a portion of the boundary of the face F_y. Finally, in $\hat{\mathcal{D}}$, we delete the edges yx_1 and yx_2. We illustrate this portion of the construction in Fig. 5.

By virtue of the construction described above, we have F_y incident with each element $x \in X$ for which there is some $s \in S$ with $x < s < y$. Now consider an element $x \in X$ so that y covers x in P. In the original drawing $\mathcal{D}$, the line segment xy intersects the circle C_y at a point p_1. Then consider a ray r emanating from x at an angle which is δ before the line segment xy. This ray intersects C_y at a point p_2. We delete the edge xy and the arc of C_y between p_1 and p_2. Then add the line segments xp_1 and xp_2. We illustrate this portion of the construction in Fig. 6.

Clearly, if δ is sufficiently small, the revised drawing $\hat{\mathcal{D}}$ is a drawing without edge crossings of a planar multigraph. We conclude that P is isomorphic to a subposet of the vertex-edge-face poset determined by $\hat{\mathcal{D}}$. With this observation, the proof of the claim is complete and Theorem 1.7 follows.

3 The Proofs—Part 2

Our proofs for Theorems 1.8 and 1.9 will require the following elementary concepts and results developed by Trotter and Moore in [16]. Recall that a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of a poset P is called a *realizer* of P

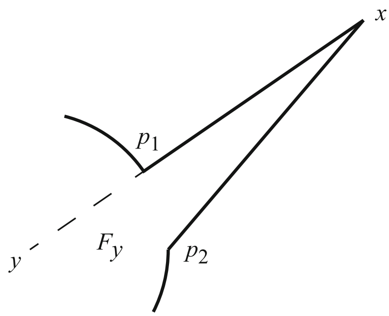

Fig. 6 Expanding the face—case 2

if $\cap\mathcal{R} = P$, i.e, for every ordered incomparable pair (x, y), there is some i for which $x > y$ in L_i. For this reason, the concept of dimension can be defined in terms of reversing ordered incomparable pairs.

A set S of ordered incomparable pairs in a poset P is *reversible* if there exists a linear extension L of P with $x > y$ in L for every $(x, y) \in S$. When P is not a chain, the dimension of P is the least t for which the set S of all ordered incomparable pairs in P can be partitioned as $S = S_1 \cup S_2 \cup \cdots \cup S_t$ with S_i reversible for each $i = 1, 2, \ldots, t$.

When $k \geq 2$, a set $\{(a_i, b_i) : 1 \leq i \leq k\}$ of ordered incomparable pairs is called an *alternating cycle* when $a_i \leq b_{i+1}$ in P for each $i = 1, 2, \ldots, k$. Of course, subscripts are interpreted cyclically in this definition. An alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$ is *strict* if for each $i = 1, 2, \ldots, k$, $a_i \leq b_j$ if and only if $j = i + 1$. In a strict alternating cycle, the sets $\{a_i : 1 \leq i \leq k\}$ and $\{b_i : 1 \leq i \leq k\}$ are k-element antichains. They need not be disjoint as there may be values of i for which $a_i = b_{i+1}$.

Alternating cycles are useful in testing whether a set of ordered incomparable pairs is reversible. The following lemma was developed in [16].

Lemma 3.1 *Let S be a set of ordered incomparable pairs in a poset P. Then the following statements are equivalent:*

1. *S is reversible.*
2. *S does not contain an alternating cycle.*
3. *S does not contain a strict alternating cycle.*

Now on to the proofs. First, let P be a poset with an outerplanar cover graph. We show that the dimension of P is at most four. Since the cover graph is outerplanar, it can be drawn as illustrated in Fig. 7. Specifically, the elements of P appear on a horizontal line, with all edges of the cover graph drawn as non-crossing arcs above the line. We let L_0 denote the left-to-right linear order $x_1 < x_2 < \cdots < x_n$ this drawing determines on the ground set of P. Of course, L_0 need not be a linear extension of P.

Now let S denote the set of all ordered pairs (x, y) with x incomparable to y in P. We define a partition $S = S_1 \cup S_2 \cup S_3 \cup S_4$ so that S_i is reversible for each $i = 1, 2, 3, 4$.

1. If (x, y) is an ordered incomparable pair with $x < y$ in L_0, assign (x, y) to S_1 when there exists a point z with $z < x$ in L_0 and $z < y$ in P; when no such point exists, assign (x, y) to S_2.

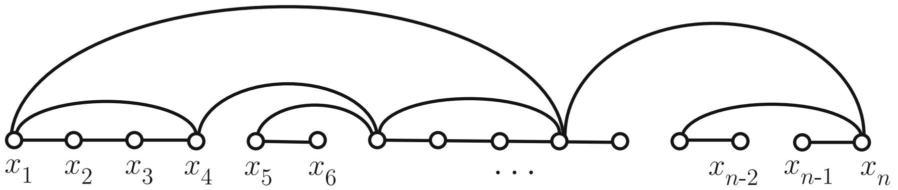

Fig. 7 A poset with an outerplanar cover graph

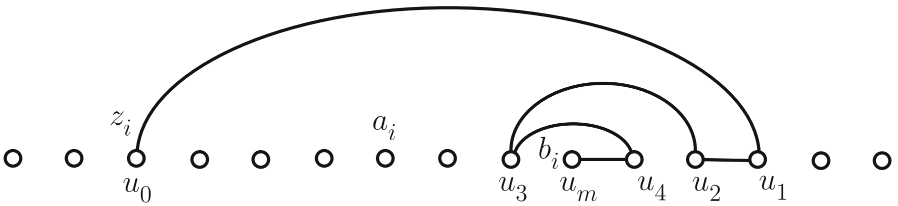

Fig. 8 Path forms a barrier

2. If (x, y) is an ordered incomparable pair with $x > y$ in L_0, assign (x, y) to S_3 when there exists a point w with $w > x$ in L_0 and $w < y$ in P; when no such point exists, assign (x, y) to S_4.

Claim 1 *S_1 and S_3 are reversible.*

Proof We give the argument for S_1, noting that the two cases are symmetric. Suppose to the contrary that S_1 is not reversible. Then there is some integer $k \geq 2$ for which S_1 contains a strict alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$. The rule used to assign (a_i, b_i) to S_1 requires that there exists a point z_i with $z_i < a_i < b_i$ in L_0 and $z_i < b_i$ in P. Then consider a chain $z_i = u_0 < u_1 < \cdots < u_m = b_i$ with u_j covered by u_{j+1} for each $j = 0, 1, \ldots, m-1$. This chain forms a "barrier" and forces (see Fig. 8) any point y with $y \geq a_i$ in P and y incomparable to b_i in P to satisfy $z_i < y < b_i$ in L_0. In fact, if u_α and u_β are points from this chain with $u_\alpha < a_i < u_\beta$ in L_0, then any point y with $a \leq y$ in P and y incomparable to b_i satisfies $u_\alpha < y < u_\beta$ in L_0. In particular, it forces $z_i < b_{i+1} < b_i$ in L_0. However, the inequality $b_{i+1} < b_i$ in L_0 cannot hold for all i. This completes the proof of the claim.

Claim 2 *S_2 and S_4 are reversible.*

Proof Again, we give the argument for S_2, noting that the two cases are symmetric. Suppose to the contrary that S_2 is not reversible. Then there is some integer $k \geq 2$ for which S_2 contains a strict alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$. Then for each $i = 1, 2, \ldots, k$, the fact that (a_i, b_i) has been assigned to S_2 requires $a_i < b_i$ in L_0. If $a_{i-1} = b_i$, we conclude that $a_i < a_{i-1}$ in L_0. On the other hand, if $a_{i-1} < b_i$, then we must have $a_i < a_{i-1}$ in L_0; otherwise (a_i, b_i) would be assigned to S_1. We conclude that in both cases, $a_i < a_{i-1}$ in L_0. Clearly, this statement cannot hold for all i. The contradiction completes the proof of the claim and Theorem 1.8 as well.

3.1 The Height 2 Case

Next, we consider the case where P has height 2 and show that the dimension of P is at most 3. The argument is very simple. Under the assumption that the height of P is 2, we claim that the set $S_1 \cup S_3$, as specified in the proof of Theorem 1.8 is reversible. Suppose to the contrary that $S_1 \cup S_3$ is not reversible. Then there is some integer $k \geq 2$ for which $S_1 \cup S_3$ contains a strict alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$.

For each $i = 1, 2, \ldots, k$, let $I(b_i)$ be the (possibly degenerate) closed interval $[p, q]$ where p is the least integer for which $x_p \leq b_i$ in P and q is the greatest integer for which $x_q \leq b_i$.

Claim 3 *For each* $i = 1, 2, \ldots, k$, $I(b_{i+1}) \subsetneq I(b_i)$.

Proof Let i be any integer with $1 \leq i \leq k$ and let $I(b_i) = [p, q]$. Suppose first that $(a_i, b_i) \in S_1$. Then there exists a point z_i with $z_i < a_i < b_i$ in L_0 and $z_i < b_i$ in P. Since P has height 2, we know that z_i is covered by b_i in P. Therefore as in the proof of Theorem 1.8, we must have $z_i < b_{i+1} < b_i$ in L_0.

Noting that $x_p \leq z_i$ in L_0, if $I(b_{i+1}) = [p', q']$, then it is clear that $p \leq p'$ and $q' < q$. This implies that $I(b_{i+1}) \subsetneq I(b_i)$. The argument when $(a_i, b_i) \in S_3$ is dual. This completes the proof of the claim, and since the set of ordered incomparable pairs can be partitioned into three reversible sets, we have shown that the dimension of P is at most 3.

4 A Construction for Theorem 1.8

Here we show that Theorem 1.8 is best possible by constructing a height 3 poset with an outerplanar cover graph for which $\dim(P) = 4$.

We define for each $n \geq 1$, a height 3 poset P_n which has an outerplanar cover graph. P_n has $3n + 1$ points:

$$\{x\} \cup \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\}.$$

For each $i = 1, 2, \ldots, n$, x covers a_i, x is covered by c_i, a_i is covered by b_i and b_i is covered by c_i. We illustrate this definition in Fig. 9, where we show a drawing of the order diagram for P_4.

Claim 4 *If* $n \geq 17$, $\dim(P_n) = 4$.

Proof Suppose to the contrary that $n \geq 17$ and $\dim(P) \leq 3$. Let $\{L_1, L_2, L_3\}$ be a realizer of P_n and consider the restriction of these three linear orders to the set $\{b_1, b_2, \ldots, b_{17}\}$. Recall the classic theorem of Erdős and Szekeres [5] which asserts that given any sequence of $m^2 + 1$ distinct integers, either there is an increasing subsequence of length $m + 1$ or there is a decreasing subsequence of length $m + 1$. By induction, it follows that if $s \geq 2$ and $L_1, L_2, \ldots, L_{s+1}$ are linear orders on a set X of size $2^{2^s} + 1$, then there are three elements x, y and z in X so that for each $j = 1, 2, \ldots, s + 1$, either $x < y < z$ or $z < y < x$ in L_j.

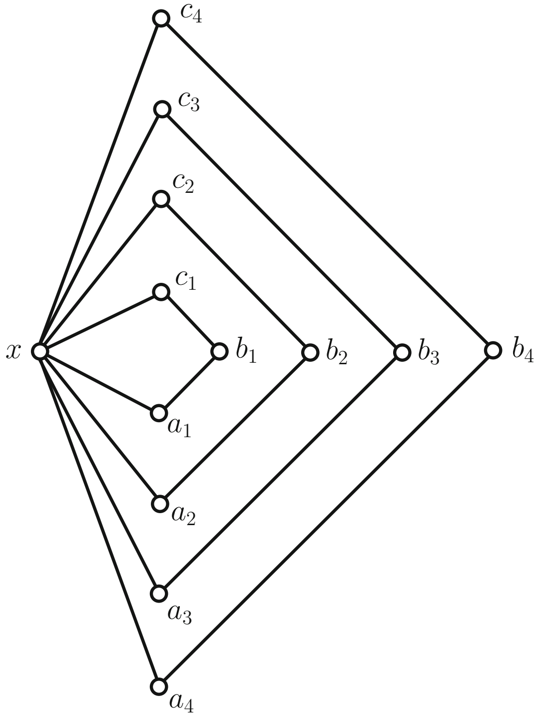

Fig. 9 A poset with outerplanar cover graph

Applying this result to our poset, with $s = 2$, and noting that $17 = 2^4 + 1$, we see that there is a 3-element subset $B = \{b_{i_1}, b_{i_2}, b_{i_3}\}$ so that for each $j = 1, 2, 3$, either $b_{i_1} < b_{i_2} < b_{i_3}$ in L_j or $b_{i_3} < b_{i_2} < b_{i_1}$ in L_j. Since these three elements form an antichain, we may assume that $b_{i_1} < b_{i_2} < b_{i_3}$ in both L_1 and L_2, while $b_{i_3} < b_{i_2} < b_{i_1}$ in L_3. Since $a_{i_1} < b_{i_1}$ and $b_{i_3} < c_{i_3}$ in P_n, it follows that $a_{i_1} < b_{i_2} < c_{i_3}$ in both L_1 and L_2. Since b_{i_2} is incomparable to both a_{i_1} and c_{i_3}, this requires $c_{i_3} < b_{i_2} < a_{i_1}$ in L_3. This is a contradiction since $a_{i_1} < c_{i_3}$ in P_n. With this, the proof of the claim is complete, and we have shown that Theorem 1.8 is best possible.

5 Open Problems

Kelly's construction leads to the following question. Is it true that for every $n \geq 3$, there exists an integer t_n so that if P is a poset with a planar cover graph and $\dim(P) > t_n$, then P contains the standard example S_n as a subposet? Here, you can also ask whether this holds under the stronger condition when P has a planar order diagram.

It is also of interest to determine for each $n \geq 3$, the least integer m_n for which there exists a poset P with m_n points for which $\dim(P) \geq n$ and P has a planar cover graph. Of course $m_3 = 6$ and $m_4 = 8$, and $m_5 \geq 12$. Again, this question can be asked for posets with planar order diagrams.

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