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Dimension and Matchings in Comparability and Incomparability Graphs

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Abstract We develop some new inequalities for the dimension of a finite poset. These inequalities are then used to bound dimension in terms of the maximum size of matchings. We prove that if the dimension of P is d and $d \geq 3$, then there is a matching of size d in the comparability graph of P. There is no analogue of this result for cover graphs, as we show that there is a poset P of dimension d for which the maximum matching in the cover graph of P has size $O(\log d)$. On the other hand, there is a dual result in which the role of chains and antichains is reversed, as we show that there is also a matching of size d in the incomparability graph of P. The proof of the result for comparability graphs has elements in common with Perles' proof of Dilworth's theorem. Either result has the following theorem of Hiraguchi as an immediate corollary: $\dim(P) \leq |P|/2$ when $|P| \geq 4$.

Keywords Matching · Dimension

1 Introduction

We assume that the reader is familiar with basic notation and terminology for graphs. In particular, a *matching* $\mathcal{M}$ in a graph G is a set of edges in G no two of which have a common endpoint. The *size* of a matching is the number of edges.

We also assume that the reader is familiar with basic notation and terminology for partially ordered sets (here we use the short term poset), including: minimal and maximal

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elements; chains and antichains; height and width; linear extensions; and order diagrams. Trotter's monograph [28] and survey article [29] remain good sources of additional background material, as are the recent papers [23], [9] and [15] on dimension theory.

When P is a poset, the *comparability graph* G_P of P is the graph whose vertex set is the ground set of P with $\{u, v\}$ an edge in G_P if and only if u and v are distinct comparable points in P. The *incomparability graph* H_P of P is just the complement of G_P, i.e. $\{x, y\}$ is an edge in H_P if and only if x and y are distinct incomparable points in P.

The *dimension* of a poset P, denoted $\dim(P)$, is the least positive integer d for which there is a family $\mathcal{R} = \{L_1, L_2, \ldots, L_d\}$ of linear extensions of P so that $x \leq y$ in P if and only if $x \leq y$ in L_i for each $i \in \{1, 2, \ldots, d\}$.

Our principal theorem bounds the dimension of a poset P by the maximum size of size of matchings in G_P and H_P.

Theorem 1.1 *Let P be a poset and let G_P and H_P be, respectively, the comparability graph and the incomparability graph of P. If* $\dim(P) = d \geq 3$, *then there is a matching of size d in G_P, and there is a matching of size d in H_P.*

As our primary focus is on the combinatorial properties of posets, we define a *chain matching* in a poset P as a family of pairwise disjoint 2-element chains in P. Dually, an *antichain matching* in P is a family of pairwise disjoint 2-element antichains in P. As the statements require separate proofs, we elect to restate Theorem 1.1 as two theorems, one for chain matchings and the other for antichain matchings.

Theorem 1.2 [Chain Matching Theorem] *Let P be a poset. If* $\dim(P) = d \geq 3$, *then P has a chain matching of size d.*

Theorem 1.3 [Antichain Matching Theorem] *Let P be a poset. If* $\dim(P) = d \geq 3$, *then P has an antichain matching of size d.*

Notice that these statements in Theorem 1.2 and Theorem 1.3 are not true when $d \leq 2$.

We comment that there are other well-known instances of combinatorial problems with analogous statements for chains and antichains. For starters, we have Dilworth's classic theorem [6] and the dual statement due to Mirsky [20].

Theorem 1.4 [Dilworth] *A poset of width w can be partitioned into w chains.*

Theorem 1.5 [Mirsky] *A poset of height h can be partitioned into h antichains.*

Second, we have the considerable strengthening of Dilworth's theorem due to Greene and Kleitman [11], with the dual result due to Greene [10].

Theorem 1.6 [Greene-Kleitman] *Let P be a poset. Then for every $k \geq 1$, there is a chain partition $P = C_1 \cup C_2 \cup \cdots \cup C_t$ so that for $i = k$ and $i = k + 1$, the maximum size of a subposet Q of P with* $\text{height}(Q) \leq i$ *is* $\sum_{j=1}^{t} \text{Min}\{i, |C_j|\}$.

Theorem 1.7 [Greene] *Let P be a poset. Then for every $k \geq 1$, there is an antichain partition $P = A_1 \cup A_2 \cup \cdots \cup A_s$ so that for $i = k$ and $i = k + 1$, the maximum size of a subposet Q of P with* $\text{width}(Q) \leq i$ *is* $\sum_{j=1}^{s} \text{Min}\{i, |A_j|\}$.

More recently, we have the following theorem proved by Duffus and Sands [7], with the dual result due to Howard and Trotter [14].

Theorem 1.8 [Duffus-Sands] *Let n and k be integers with $n \geq k \geq 3$, and let P be a poset. If $n \leq |C| \leq n + (n-k)/(k-2)$, for every maximal chain C in P, then P has k pairwise disjoint maximal antichains.*

Theorem 1.9 [Howard-Trotter] *Let n and k be integers with $n \geq k \geq 3$, and let P be a poset. If $n \leq |A| \leq n + (n-k)/(k-2)$ for every maximal antichain A in P, then P has k pairwise disjoint maximal chains.*

As is the case with Dilworth's theorem and its dual, our theorem for chain matchings is more challenging than the result for antichain matchings, and we do not know of any "perfect graph" underpinning that makes the two results equivalent. We will also show that our chain matching theorem cannot be extended to cover graphs by showing that there is a d-dimensional poset for which the maximum size of a matching in the cover graph has size $O(\log d)$.

The remainder of this paper is organized as follows. In the next section, we provide essential notation, terminology and background material. Most of this will be familiar to readers who have some experience with combinatorial problems for posets. In Section 3, we provide a brief sketch of related work which serves to motivate the research reported here. In Section 4, we prove three new inequalities for dimension. The first is elementary, but the second and third are more substantive, and they are the key ingredients of the proofs for our matching theorems. In Section 5, we discuss chain matchings and prove the chain matching theorem. At the close of this section, we explain why there is no analogue for matchings in the cover graph. In Section 6, we prove the antichain matching theorem, and we close in Section 7 with a brief discussion of some open problems.

2 Notation, Terminology and Preliminary Material

When P is a poset, we use $|P|$ to denote the number of elements in the ground set of P. Subposets of P are identified just by specifying their ground sets. For example, if x and y are distinct elements of P, $P - \{x, y\}$ is just the subposet obtained when x and y are removed from P. When S is a subposet of P and L is a linear extension of P, $L(S)$ denotes the restriction of L to S.

Define *blocks* to be disjoint subsets of vertex set of P. Blocks will be used to define a linear extension of P. In Fig. 1, we show a poset P. For this poset, define blocks $B_1 = \{a, e, f, h, j\}$, $B_2 = \{b, g, k\}$, $B_3 = \{i, l\}$ and $B_4 = \{c, d\}$. Then set $L_2 = [j < a < b < k < h < g]$ and $L_4 = [b < k < d < c]$. In this paper, we use notation such as

$$L = [B_1 < u_1 < L_2(B_2) < u_2 < B_3 < u_3 < L_4(B_4)]$$

to define a linear extension of P. Technically speaking, we have not precisely defined a particular linear extension, since we intend that the choice of the extension on the blocks B_1 and B_3 is arbitrary.

When a linear extension L has been defined in this manner, and we subsequently specify that $L' = L(S)$, we intend that suitable choices have been made for L. In particular, if A is an antichain and we say that $L' = L^*(A)$, then for every distinct pair $a, a' \in A$, if $a < a'$ in L, then $a' < a$ in L'.

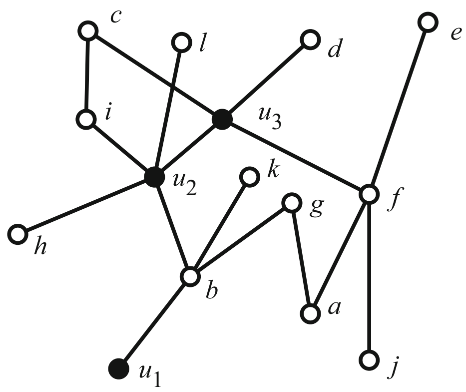

Fig. 1 Defining Linear Extensions Using Blocks

The *width* of a poset P, denoted width(P) is the maximum size of an antichain in P, while Min(P) and Max(P) denote, respectively, the set of minimal elements and the set of maximal elements in P. When $x \in P$, we write $D(x)$ for the set $\{y \in P : y < x \text{ in } P\}$ while $U(x)$ is the set $\{y \in P : y > x \text{ in } P\}$. When A is a maximal antichain in P, $D(A)$ consists of all $x \in P$ for which there is some $a \in A$ with $x < a$ in P. Dually, $U(A)$ consists of all $x \in P$ for which there is some $a \in A$ with $x > a$ in P. Evidently, $D(A) \cap U(A) = \emptyset$.

A non-empty subset D of a poset P is called a *down set* if $x \in D$ whenever $y \in D$ and $x < y$ in P. Dually, a subposet U of P is called an *up set* if $y \in U$ whenever $x \in U$ and $x < y$ in P. Note that D is a down set if and only if $U = P - D$ is an upset. Also note that when $x \in P$, $D(x)$ is a down set and $U(x)$ is an upset. Also, when A is a maximal antichain in P, both $D(A)$ and $A \cup D(A)$ are down sets, while $U(A)$ and $A \cup U(A)$ are up sets.

A non-empty family $\mathcal{F}$ of linear extensions of a poset P is called a *realizer* of P when $x \le y$ in P if and only if $x \le y$ in L for every $L \in \mathcal{F}$. The notation $u \parallel v$ in P means u and v are incomparable points in P. Inc(P) denotes the set of all ordered pairs (u, v) where $u \parallel v$ in P. Considered as a binary relation, Inc(P) is symmetric. Accordingly, a non-empty family $\mathcal{F}$ of linear extensions of P is a realizer if and only if for every $(u, v) \in \text{Inc}(P)$, there is some $L \in \mathcal{F}$ with $u > v$ in L.

An incomparable pair $(u, v) \in \text{Inc}(P)$ is called a *critical pair* if (1) $z < v$ in P whenever $z < u$ in P; and (2) $w > u$ in P whenever $w > v$ in P. We denote the set of all critical pairs in P by Crit(P). A non-empty family $\mathcal{F}$ of linear extensions of a poset P is a realizer of P if and only if for every critical pair $(u, v) \in \text{Crit}(P)$, there is some $L \in \mathcal{F}$ with $u > v$ in L.

It is clear that dimension is *monotone*, i.e., if Q is a subposet of P, then $\dim(Q) \le \dim(P)$. The next result, due to Hiraguchi [12], is fundamental to our subject. We include the elementary proof as a precursor of arguments to follow.

Theorem 2.1 *Let P be a poset with at least two points. If $x \in P$, then* $\dim(P) \le 1 + \dim(P - \{x\})$.

Proof Let $\dim(P - \{x\}) = t$ and let $\{L_1, L_2, \ldots, L_t\}$ be a realizer of $P - \{x\}$. For each $i \in \{1, 2, \ldots, t-1\}$, let R_i be any linear extension of P so that $R_i(P - \{x\}) = L_i$. Then set:

$$R_t = [L_t(D(x)) < x < L_t(P - D(x))] \quad \text{and} \\ R_{t+1} = [L_t(P - U(x)) < x < L_t(U(x))]. \tag{1}$$

Clearly, $\{R_1, R_2, \ldots, R_{t+1}\}$ is a realizer of P. □

We view the preceding theorem as asserting that dimension is "continuous", i.e., small changes in the poset can only make small changes in dimension.

We use the notation P^* for the *dual* of P, i.e., P^* has the same ground set as P with $x > y$ in P^* if and only if $x < y$ in P. Note that $\dim(P) = \dim(P^*)$. Also note that a chain matching in P is also a chain matching in P^*. The same statement holds for antichain matchings. So not surprisingly, duality will play an important role in arguments to follow.

A poset P (with at least two points) is *irreducible* when $\dim(P - \{x\}) < \dim(P)$ for every x in P. An irreducible poset P is d-irreducible if $\dim(P) = d$. The only 2-irreducible poset is a 2-element antichain. A full listing of all 3-irreducible posets has been assembled by Kelly [16] and by Trotter and Moore [33]. These posets are illustrated in Figs. 2 and 3. The posets shown in the first figure constitute seven infinite families, while the posets shown in the second figure are "miscellaneous" examples. In cases where a 3-irreducible poset is not self-dual, only one of the two instances is included in these figures.

It is a straightforward exercise to show that every 3-irreducible poset has both a chain matching of size 3 and an antichain matching of size 3. As a consequence, when presenting the proofs[1] of our two matching theorems, we will restrict our attention to posets with dimension at least 4.

When $n \geq 8$, inspection of the posets illustrated in Figs. 2 and 3 shows that the number of 3-irreducible posets on n points is at most 7. On the other hand, for each $d \geq 4$, Trotter and Ross [34] showed that there is a constant $c_d > 1$ and an integer n_d so that for $n \geq n_d$, the number of d-irreducible posets on n points is more than $c_d^{n^2}$. As a consequence, it may seem that we have not accomplished much by our short proofs of the Theorems 1.2 and 1.3 when $\dim(P) = 3$, as apparently this is just the easy case with the real work yet to be done.

A subposet Y of a poset P is said to be *autonomous* if for every $u_0 \in Y$, the following two statements hold:

(1) If $v \in P - Y$ and $v < u_0$, then $v < u$ for every $u \in Y$; and
(2) If $v \in P - Y$ and $v > u_0$, then $v > u$ for every $u \in Y$.

Any singleton set $\{u\}$ in P is autonomous, as is the entire poset P. Hiraguchi [12] noted that if P is a poset and Y is an autonomous subposet of P with $2 \leq |Y| < P$ and $u_0 \in Y$, then $\dim(P) = \mathrm{Max}\{\dim(Y), \dim(P - (Y - \{u_0\}))\}$. As a consequence such a poset is not irreducible. We say a poset P is *indecomposable* when there is no autonomous subposet Y of P with $2 \leq |Y| < |P|$, noting that irreducible posets are indecomposable.

The definition of dimension is due to Dushnik and Miller [8], who also gave the following construction.

For each $d \geq 2$, the *standard example* S_d is the poset of height 2 with minimal elements $\{a_1, a_2, \ldots, a_d\}$, maximal elements $\{b_1, b_2, \ldots, b_d\}$ and ordering $a_i < b_j$ in S_d if and only if $1 \leq i, j \leq d$ and $i \neq j$. We claim that $\dim(S_d) = d$ for all $d \geq 2$. To see this, first note that $\mathrm{Crit}(P) = \{(a_i, b_i) : 1 \leq i \leq d\}$. The fact that $\dim(S_d) \geq d$ follows from the simple fact that if L is a linear extension of S_d, there can be at most one value of i for which $a_i > b_i$ in L. This shows $\dim(S_d) \geq d$.

[1]It is of course possible to handle the case of 3-dimensional posets directly—without full knowledge of the list of 3-irreducible posets using the arguments for the general case we present here coupled with structural results developed in [19] and [27]. However, this approach still requires some work and we do not see much gain to the effort.

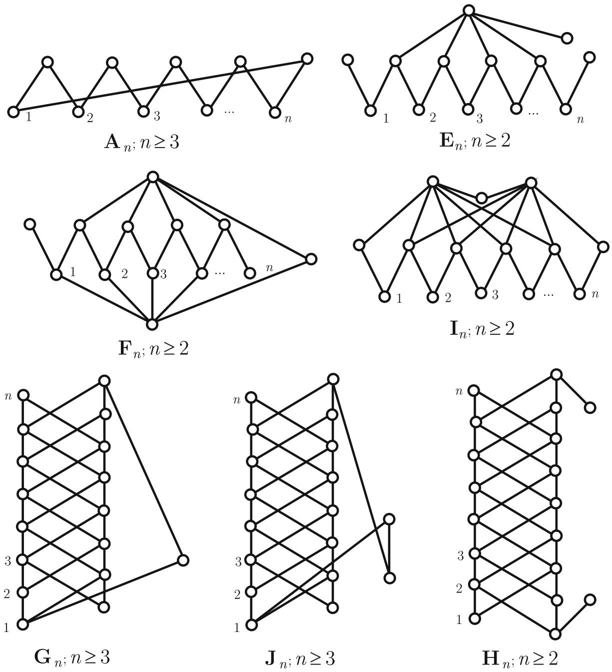

Fig. 2 Families of 3-Irreducible Posets

On the other hand, for each $i \in \{1, 2, \dots, d\}$, there is a linear extension L_i of S_d with $a_i > b_i$ in L_i. This shows $\dim(S_d) \leq d$. Trivially, S_d has both a chain matching of size d and an antichain matching of size d.

3 Motivation

3.1 Hiraguchi's Inequality

In 1951, Hiraguchi [12] proved the following key lemma. A proof is provided here as the basic idea is essential for several of our proofs to follow.

Lemma 3.1 *If C is a chain in a poset P, then there exists a linear extension L of P so that $u > v$ in L for every $(u, v) \in \mathrm{Inc}(P)$ with $u \in C$.*

Proof Label the points in the chain C as $u_1 < u_2 < \cdots < u_s$. Then partition the points in $P - C$ into blocks $B_1, B_2, \dots, B_{s+1}$ where an element v from $P - C$ belongs to B_1 if $v \not> u_1$; otherwise v belongs to B_{j+1} where j is the largest integer so that $v > u_j$. Then set

$$L = [B_1 < u_1 < B_2 < u_2 < B_3 < \cdots < B_s < u_s < B_{s+1}].$$

It is clear that L satisfies the requirement of the lemma. □

Returning to the poset shown in Fig. 1, we see that the previous discussion regarding this poset serves to illustrate the application of Lemma 3.1 to the chain $C = \{u_1, u_2, u_3\}$.

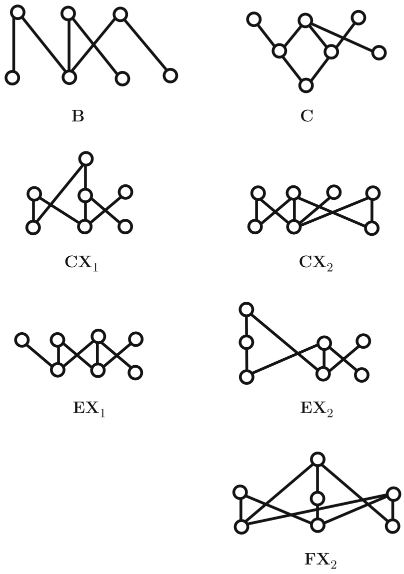

Fig. 3 Miscellaneous Examples of 3-Irreducible Posets

However, as we will use this technique later, we elected to also specify how the linear extension L would order blocks B_2 and B_4.

It is useful to view the construction in Lemma 3.1 as pushing the chain C "up" while forcing all elements of $P - C$ "down" as low as possible relative to C, so we say that the resulting linear extension puts C "over" $P - C$. Also, we comment that there is a dual form of this lemma, i.e., there is a linear extension L with $u > v$ in L whenever $(u, v) \in \mathrm{Inc}(P)$ and $v \in C$. This extension puts $P - C$ "over" C.

Hiraguchi noted in [12] that when Lemma 3.1 is used for each chain in a minimum chain cover provided by Dilworth's theorem, we have the following upper bound.

Lemma 3.2 *If P is a poset, then* $\dim(P) \le \mathrm{width}(P)$.

The principal result in [12] is the following upper bound.

Theorem 3.3 *[Hiraguchi's Inequality] If P is a poset, then* $\dim(P) \le |P|/2$, *when* $|P| \ge 4$.

Hiraguchi's original 1951 proof of this inequality was relatively complicated. In 1955, he gave an updated and somewhat streamlined proof [13], and in 1974, Bogart gave a more polished version [1]. Subsequently, Kimble [19] and Trotter [25] discovered the following inequality, and this result in combination with Lemma 3.2 yields an elegant proof of Hiraguchi's theorem.

Theorem 3.4 *If A is an antichain in a poset P, then* $\dim(P) \le \mathrm{Max}\{2, |P - A|\}$.

We comment that there is an elementary inductive proof of Theorem 3.4 with the base case being $|P - A| = 2$. Here, if the inequality fails, there is a 3-irreducible (and therefore indecomposable) poset, having width at least 3 and consisting of an antichain plus at most 2 other points. However, it is an easy exercise to show that there are only two (up to duality) posets satisfying these requirements. These two posets are shown in Fig. 4. A second elementary exercise is to show that both of these posets have dimension 2. The theorem then follows by induction, using Theorem 2.1 to show that the dimension decreases by at most one each time an element of $P - A$ is removed.

Although this was not the primary motivation for our research, we note that Hiraguchi's inequality is an immediate corollary of both our chain matching and our antichain matching theorems.

3.2 The Removable Pair Conjecture

When $|P| \geq 3$, a distinct pair $\{x, y\}$ is called a *removable pair* if $\dim(P) \leq 1 + \dim(P - \{x, y\})$. When $\dim(P) \leq 3$, every distinct pair of points is a removable pair. However, when $\dim(P) \geq 4$, no comparable pair in the standard example S_d is a removable pair. Also, among the incomparable pairs, only the critical pairs are removable.

Conjecture 3.5 *[The Removable Pair Conjecture] If P is a poset and $|P| \geq 3$, then P has a removable pair, i.e., there exist distinct elements x and y in P so that* $\dim(P) \leq 1 + \dim(P - \{x, y\})$.

Although its origins have been obscured with the passage of time, the Removable Pair Conjecture (RPC) has been investigated by researchers for more than 60 years. Anyone reading Hiraguchi's early 1951 and 1955 papers will certainly be convinced that he tried hard to prove the RPC, even though it is not stated explicitly as a problem or conjecture in either of these papers. Nevertheless, the first published reference to the RPC seems to be in a 1975 paper of Trotter [25]. We also note that the RPC was one of the "Unsolved Problems," assembled by the Editorial Board of *Order* and appearing for more than 10 years in each issue of the journal (see [17]).

If the RPC holds, then a simple inductive proof of Hiraguchi's theorem could be obtained just by establishing the base case: $\dim(P) \leq 2$ when $|P| \leq 5$. This is an easy exercise, as a counterexample would have to be an indecomposable poset with width at least 3. This is essentially the same exercise discussed previously, as there are only two such posets, the ones shown in Fig. 4, and both have dimension 2.

Here are two of many conditions which guarantee that a pair is removable. The first result is due to Hiraguchi [12] while the second is part of the folklore of the subject, although it is implicit in Theorem 7.4 as presented in [30]. We provide a short proof since this result will be quite useful to us in proving our main theorem for antichain matchings.

Theorem 3.6 *Let P be a poset with $|P| \geq 3$, let a and b be distinct points in P with $a \in \mathrm{Min}(P)$ and $b \in \mathrm{Max}(P)$. If $a \parallel b$ in P, then $\{a, b\}$ is a removable pair.*

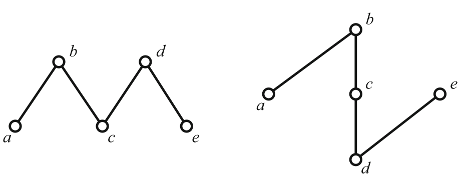

Fig. 4 Two Small Indecomposable Posets with Width at least 3

Theorem 3.7 *Let P be a poset with $|P| \geq 3$ and let a and a' be distinct minimal elements in P, If $U(a) \subseteq U(a')$, then $\{a, a'\}$ is a removable pair.*

Proof Let $Q = P - \{a, a'\}$, let $t = \dim(Q)$ and let $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of Q. For each $i \in \{1, 2, \ldots, t\}$, let $R_i = [a < a' < L_i]$. Then set

$$R_{t+1} = [Q - U(a') < a' < U(a') - U(a) < a < U(a)].$$

Evidently, $\{R_1, R_2, \ldots, R_t, R_{t+1}\}$ is a realizer of P so that $\dim(P) \leq 1 + \dim(Q)$. □

Several strong versions of the RPC conjecture have been offered, and some of these have been disproved. Bogart [2] suggested that a removable pair $\{x, y\}$ could always be found among the elements of $\mathrm{Max}(P) \cup \mathrm{Min}(P)$. This was disproved by Trotter and Monroe [32] who constructed for each $t \geq 1$, a poset P_t so that $|P| = (3t+1)^2 + (6t+2)$, $\dim(P) = 4t + 2$, and $\dim(P - \{x, y\}) = 4t$ for every distinct pair $x, y \in \mathrm{Max}(P) \cup \mathrm{Min}(P)$. Subsequently, Bogart and Trotter conjectured [31] that every critical pair was removable, a result motivated by their work on interval dimension. However, this form of the RPC was disproved by Reuter [22].

Historically, it should be noted that in [22], Reuter gives a diagram for his counterexample, a 4-dimensional poset with 14 points. Reuter also remarks that Trotter relayed to him that his counterexample is not minimal and that a smaller counterexample can be obtained. Subsequently, Kierstead and Trotter [18] gave a general construction which shows that for every $d \geq 5$, there is a poset P_d containing a critical pair (x, y) so that $\dim(P) = d$ and $\dim(P - \{x, y\}) = d - 2$. In [18], Kierstead and Trotter referenced Reuter and provided a diagram intended to be the appropriate 12-element poset extracted from Reuter's original counterexample. Inexplicably the diagram is incorrectly drawn in [18]. Subsequently, the incorrect figure also appears in Trotter's monograph [28] and at least two journal papers. A corrected drawing is given here in Fig. 5 alongside the more general construction given in [18].

We should comment that it is still open to determine whether every poset P with $|P| \geq 3$ contains *some* critical pair which is removable. Were this to be true, then a simple proof of our main result for antichain matchings would be produced. On the other hand, it is easy to construct examples (not just the standard examples) of posets where some comparable pairs are not removable, e.g., the max-min pairs in the examples given in [32]. So it will be a much greater challenge to prove our chain matching theorem with this approach.

Although the RPC remains open, there are some useful removal theorems which are more general. We say that two pairwise disjoint chains C and C' in a poset P are *incomparable* when $x \parallel y$ in P for every $x \in C$ and $y \in C'$. The following results are

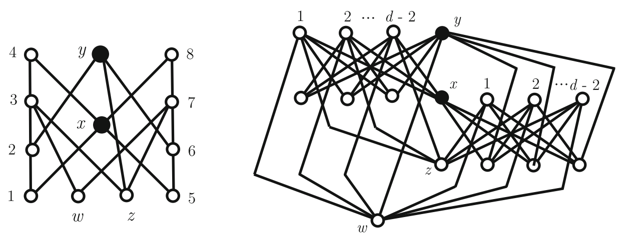

Fig. 5 Critical Pairs Need Not be Removable

due to Hiraguchi [12]. The first is a straightforward generalization of Lemma 3.1 to two incomparable chains. The second is an immediate corollary of the first.

Lemma 3.8 *Let C_1 and C_2 be non-empty incomparable chains in a poset P. Then there exist linear extensions L and L' of P so that*

(1) *C_1 is over $P - C_1$ and $P - C_2$ is over C_2 in L, and*
(2) *C_2 is over $P - C_2$ and $P - C_1$ is over C_1 in L'.*

Lemma 3.9 *Let C_1 and C_2 be non-empty incomparable chains in a poset P. If there is some point of P which does not belong to either chain, then $\dim(P) \leq 2 + \dim(P - (C_1 \cup C_2))$.*

3.3 Related Inequalities

In [25], Trotter proved the following inequality.

Theorem 3.10 *Let P be a poset which is not an antichain, and let $w = \text{width}(P - \text{Max}(P))$. Then $\dim(P) \leq w + 1$.*

This inequality is tight for all $w \geq 1$, and we refer the reader to [25] for details. In this same paper, Trotter also proved the following inequality.

Theorem 3.11 *Let A be a maximal antichain in a poset P which is not a chain, and let $w = \text{width}(P - A)$. Then $\dim(P) \leq 2w + 1$.*

It is more complicated to show that this inequality is tight, and the argument is deferred to a separate paper [24]. In the next section, we will provide strengthenings of both results.

4 Three New Inequalities for Dimension

4.1 New Inequalities for Dimension

In this paper, we will need a straightforward extension of Theorem 3.11.

Theorem 4.1 *Let D be a non-empty down set in a poset P such that the up set $U = P - D$ is also non-empty. If $\dim(D) = t$ and $\text{width}(U) = w$, then $\dim(P) \leq t + w$.*

Proof Let $\text{width}(U) = w$ and let $U = C_1 \cup C_2 \cup \cdots \cup C_w$. Let $\dim(D) = t$ and let $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of D. For each $i \in \{1, 2, \ldots, t\}$, let $R_i = [L_i < U]$. Then for each $j \in \{1, 2, \ldots, w\}$, let R_{t+j} be a linear extension which puts $P - C_j$ over C_j. Clearly, the family $\{R_1, R_2, \ldots, R_{t+w}\}$ of linear extensions of P is a realizer. □

Of course, there is a dual version of the preceding theorem in which the roles of U and D are reversed. In either case, it is straightforward to modify the arguments given in [24] to show that the inequality is tight.

The next theorem provides a condition under which the inequality $\dim(P) \leq 1 + \text{width}(P - \text{Max}(P))$ in Theorem 3.10 can be improved. Although this result is somewhat technical, it is an essential ingredient of proofs to follow.

Theorem 4.2 *Let P be a poset and let $Q = P - \mathrm{Max}(P)$. If* $\mathrm{width}(Q) = w \geq 2$, *and there is a point $x \in Q$ so that* $\mathrm{width}(Q - \{x\}) = w - 1$, *then* $\dim(P) \leq w$.

Proof We argue by contradiction and assume that P is a counterexample of minimum size. Then P is $(w+1)$-irreducible.

Let $C_1 \cup C_2 \cup \cdots \cup C_{w-1} \cup \{x\}$ be a chain cover of $Q = P - \mathrm{Max}(P)$. Setting $C_w = \{x\}$, we apply Lemma 3.1 to choose for each $i \in \{1, 2, \ldots, w\}$, a linear extension L_i which puts C_i over $P - C_i$. We then modify L_{w-1} and L_w to form L'_{w-1} and L'_w and show that the family $\mathcal{F}' = \{L_1, L_2, \ldots, L_{w-2}, L'_{w-1}, L'_w\}$ must be a realizer of P.

Let $C_{w-1} = \{u_1 < u_2 < \cdots < u_s\}$. The point x must be incomparable with at least one point of C_{w-1}; else $C_{w-1} \cup \{x\}$ is a chain, $\mathrm{width}(Q) \leq w - 1$ and $\dim(P) \leq w$.

Then let $A_1 = \mathrm{Max}(P) \cap U(x)$ and $A_2 = \mathrm{Max}(P) - A_1$. Then set

$$L'_{w-1} = [L_{w-1}(P - A_2) < L^*_{w-1}(A_2)].$$

Now in forming the linear extension L_w, there are only two blocks, and to distinguish these from the blocks appearing in L_{w-1}, we denote these as S_1 and S_2. Let $S_1 = P - U(x) - \{x\}$, and $S_2 = U(x)$. Now set

$$L'_w = [L_{w-1}(S_1) < x < L_{w-1}(S_2 - A_1) < L^*_{w-1}(A_1)].$$

Since the family $\mathcal{F}' = \{L_1, L_2, \ldots, L_{w-2}, L'_{w-1}, L'_w\}$ cannot be a realizer of P, there is some critical pair $(u, v) \in \mathrm{Crit}(P)$ with $u < v$ in each of the w linear extensions in $\mathcal{F}'$. Clearly, there is no i with $1 \leq i < w - 2$ for which $u \in C_i$. Now suppose that $u \in C_{w-1}$. Since $u > v$ in L_{w-1}, we must have $v \in A_2$. This implies that $v \parallel x$ in P, so that v belongs to block S_1 in L_w. It follows that $u > v$ in L'_w.

If $u = x$, then $u > v$ in L'_w.

Finally, suppose that $u \in \mathrm{Max}(P)$. If $u \in A_1$ and $u < v$ in L'_{w-1}, then $u > v$ in L'_w. If $u \in A_2$ and $u < v$ in L'_w, then $u > v$ in L'_{w-1}. □

4.2 The Inequality of Theorem 4.2 is Tight

We show in Fig. 6 a family $\{P_d : d \geq 3\}$ of posets. Observe that the following properties hold:

(1) $\mathrm{Max}(P_d) = \{a_0, a_1, a_2, \ldots, a_d\}$.
(2) $\mathrm{width}(P_d - \mathrm{Max}(P_d)) = d - 1$.
(3) $\mathcal{M} = \{\{x_i < y_i\} : 1 \leq i \leq d - 1\}$ is a chain matching in P, but it is not maximum.

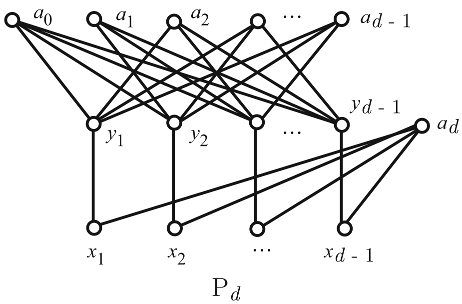

Fig. 6 Witnessing that the Inequality is Tight

We now show that $\dim(P_d) = d$ for all $d \geq 3$. Accordingly, this shows that the inequality $\dim(P) \leq 1+\text{width}(P-\text{Max}(P))$ is best possible. However, it also shows that the condition of Theorem 4.2 cannot be weakened, since the chains covering $P_d - \text{Max}(P_d)$ have size 2.

Suppose to the contrary that $\dim(P) \leq d-1$. We note that the subposet of P_d determined by the points in $\{x_i : 1 \leq i \leq d-1\} \cup \{a_i : 1 \leq i \leq d-1\}$ is isomorphic to the standard example S_{d-1}, so we must then have $\dim(P_d) = d-1$. Now let $\mathcal{F} = \{L_1, L_2, \ldots, L_{d-1}\}$ be a realizer of P_d. We may assume that the extensions in $\mathcal{F}$ have been labeled so that $x_i > a_i$ in L_i for each $i \in \{1, 2, \ldots, d-1\}$. Then for all $i, j \in \{1, 2, \ldots, d-1\}$ with $i \neq j$, we have

$$x_j < y_j < a_i < x_i < y_i < a_0 \quad \text{in } L_i.$$

Since $a_d \parallel y_j$ for $j \in \{1, 2, \ldots, d-1\}$ and $a_d > x_i$ for each $i \in \{1, 2, \ldots, d-1\}$, we must therefore have $x_i < a_d < y_i < a_0$ in L_i for each $i \in \{1, 2, \ldots, d-1\}$. This is a contradiction since $\mathcal{F}$ does not reverse the critical pair (a_d, a_0).

We leave it as an exercise to show that for each $d \geq 4$, the poset P_d is d-irreducible. However, P_3 is not irreducible, as $P_3 - \{a_1, a_2\}$ is isomorphic to the dual of the poset C shown in Fig. 3. This poset is traditionally called the "chevron".

The reader may note that for this family, we have $\{(a_i, a_0) : 1 \leq i \leq d-1\} \subset \text{Crit}(P_d)$. Furthermore, for each $j \in \{1, 2, \ldots, d-1\}$, if we add the comparabilities $a_0 > a_i$ for $1 \leq i \leq j$ to P_d, we obtain a height four poset with $\text{Max}(P) = \{a_0, a_d\}$, which also shows that the inequality in Theorem 3.10 is tight. The case when $j = d-1$ is just the construction given in [25]. We leave it as an exercise to show that all such posets are d-irreducible when $d \geq 4$.

4.3 An Inequality Involving Matchings

Here is another result which gives a condition under which the the inequality $\dim(P) \leq 1 + \text{width}(P - \text{Max}(P))$ of Theorem 3.10 can be tightened. The reader may note that this theorem does the lion's share of the work in proving that a 3-dimensional poset has a chain matching of size 3, without appeal to the list of 3-irreducible posets.

When $\mathcal{M}$ is a maximum chain matching in a poset P, we let $A(\mathcal{M})$ denote the set of all points in P which are not covered by the chains in $\mathcal{M}$. Evidently, $A(\mathcal{M})$ is an antichain.

Theorem 4.3 *Let P be a poset which is not an antichain, and let $\mathcal{M}$ be a maximum chain matching in P. If $\mathcal{M}$ has size m and $A(\mathcal{M}) \subseteq \text{Max}(P)$, then* $\dim(P) \leq \text{Max}\{2, m\}$.

Proof We argue by contradiction. We assume the conclusion of the theorem is false and choose a counterexample P with $|P|$ minimum. Let $d = \dim(P)$, noting that $d \geq 3$. Set $m = |\mathcal{M}|$. Since P is not an antichain, $m \geq 1$.

Let $Q = P - \text{Max}(P)$. Then $1 \leq \text{width}(Q) \leq m < d$. By Theorem 3.10, we conclude $m = \text{width}(Q) = d - 1$, which forces $m \geq 2$.

Claim 1. $A(\mathcal{M}) = \text{Max}(P)$.

Proof Suppose to the contrary that there is a maximal element y so that y belongs to a chain C in the matching $\mathcal{M}$. Let x be the other point in C. It follows that $P - \text{Max}(P)$ is covered by m chains, with one of the chains being the single point $\{x\}$. Now Theorem 4.2 implies that $\dim(P) \leq m$. The contradiction completes the proof of the Claim 1.

Claim 2. $\dim(P - \{u\}) = d - 1$ for every $u \in \text{Max}(P)$.

Proof Let $u \in \mathrm{Max}(P)$ and assume that $\dim(P - \{u\}) = d$. Then $\mathcal{M}$ is also a maximum matching in $P - \{u\}$, so $P - \{u\}$ would be a counterexample, contradicting our choice of P. The contradiction completes the proof of Claim 2.

Label the chains in $\mathcal{M}$ as $\{C_i = \{x_i < y_i\} : 1 \leq i \leq m\}$. Apply Lemma 3.1 for each chain C_i to obtain a linear extension L_i which puts C_i over $P - C_i$. Note that in the construction of L_i, there are three blocks which we denote $B_{i,1}$, $B_{i,2}$ and $B_{i,3}$. For each $i \in \{1, 2, \ldots, m-1\}$, we then modify L_i to form L_i' by altering the order on the three blocks as follows:

$$L_i'(B_{i,j}) = [B_{i,j} - \mathrm{Max}(P) < L_m^*(B_{i,j} \cap \mathrm{Max}(P))] \quad \text{for } j \in \{1, 2, 3\}.$$

Finally, we modify L_m to form L_m' by setting:

$$L_m'(B_{m,j}) = [B_{m,j} - \mathrm{Max}(P) < L_m(B_{m,j} \cap \mathrm{Max}(P))] \quad \text{for } j \in \{1, 2, 3\}.$$

Note that elements always remain in the same block. We simply pull the maximal elements to the top of the block and then order them in a manner that is dual to how they are ordered elsewhere.

Since P is a counterexample, the family $\mathcal{F} = \{L_i' : 1 \leq i \leq m\}$ cannot be a realizer of P, so there is a critical pair $(u, v) \in \mathrm{Crit}(P)$ for which $u < v$ in L_i' for all $i \in \{1, 2, \ldots, m\}$. Since L_i' puts C_i over $P - C_i$, this clearly forces $u \in \mathrm{Max}(P)$, and in turn this forces $v \in \mathrm{Max}(P)$.

Since $\dim(P - \{u\}) = d - 1$, we know that the maximal element u is not a minimal element of P, so $u > x_i$ in P for all $i \in \{1, 2, \ldots, m\}$. Since $(u, v) \in \mathrm{Crit}(P)$, we know that $v > x_i$ for all $i \in \{1, 2, \ldots, m\}$.

For every $i \in \{1, 2, \ldots, m\}$, since $u, v > x_i$ in P, we know that u, v belong to one of $B_{i,2}$ and $B_{i,3}$. However, if u and v are in the same block in each L_i', then in L_1' they are ordered in reverse order to how they are ordered in L_m'. So there must exist a $j \in \{1, 2, ..., m\}$ such that we have $u \in B_{j,2}$ and $v \in B_{j,3}$. This implies that $v > y_j$. However, this implies that the chain matching $\mathcal{M}$ is not maximum, since we could replace $C_j = \{x_j < y_j\}$ with $C' = \{x_j < u\}$ and $C'' = \{y_j < v\}$. The contradiction completes the proof of the theorem. $\square$

Again, the example illustrated in Fig. 6 shows that the inequality in Theorem 3.10 is tight.

5 Chain Matchings

5.1 Existence of Pure Maximum Chain Matchings

Let $\mathcal{M}$ be a maximum chain matching in a poset P. As in the preceding section, we let $A(\mathcal{M})$ denote the antichain consisting of the points in P which are not covered by chains in $\mathcal{M}$. If $A(\mathcal{M}) = \emptyset$, then $\dim(P) \leq \mathrm{width}(P) \leq m$, so our focus in this section will be on posets for which $A(\mathcal{M})$ is a non-empty antichain. In this case, we then let $\mathbb{U}(\mathcal{M})$ denote the set of all chains $C \in \mathcal{M}$ for which there is an element $a \in A(\mathcal{M})$, an integer $s \geq 1$ and a sequence $\{C_i = \{x_i < y_i\} : 1 \leq i \leq s\}$ of distinct chains in $\mathcal{M}$ so that (1) $y_1 > a$ in P, (2) $y_{i+1} > x_i$ in P whenever $1 \leq i < s$ and (3) $C = C_s$. Dually, $\mathbb{D}(\mathcal{M})$ denotes the set of all chains $C' \in \mathcal{M}$ for which there is an element $a' \in A(\mathcal{M})$, an integer $s' \geq 1$ and a sequence $\{C_i' = \{x_i' < y_i'\} : 1 \leq i \leq s'\}$ of distinct chains in $\mathcal{M}$ so that (1) $x_1' < a'$ in P, (2) $x_{i+1}' < y_i'$ in P whenever $1 \leq i < s'$ and (3) $C' = C_{s'}$.

In Fig. 7, we show a poset P and a maximum chain matching $\mathcal{M} = \{\{x_i < y_i\} : 1 \leq i \leq 6\}$. It is easy to see that:

(1) The chains $\{x_1 < y_1\}$, $\{x_2 < y_2\}$, $\{x_3 < y_3\}$ and $\{x_4 < y_4\}$ belong to $\mathbb{U}(\mathcal{M}) \cap \mathbb{D}(\mathcal{M})$.
(2) The chain $\{x_5 < y_5\}$ belongs to $\mathbb{U}(\mathcal{M})$ but not to $\mathbb{D}(\mathcal{M})$.
(3) The chain $\{x_6 < y_6\}$ belongs to $\mathbb{D}(\mathcal{M})$ but not to $\mathbb{U}(\mathcal{M})$.
(4) The chain $\{x_7 < y_7\}$ does not belong to $\mathbb{U}(\mathcal{M})$ or to $\mathbb{D}(\mathcal{M})$.

We say that a maximum chain matching $\mathcal{M}$ in P is *pure* if $\mathbb{U}(\mathcal{M}) \cap \mathbb{D}(\mathcal{M}) = \emptyset$. A maximum chain matching $\mathcal{M}$ with $A(\mathcal{M}) \subseteq \text{Max}(P)$ is pure since $\mathbb{U}(\mathcal{M}) = \emptyset$. Dually, a maximum chain matching $\mathcal{M}$ with $A(\mathcal{M}) \subseteq \text{Min}(P)$ is pure. On the other hand, there are posets having no maximum chain matching $\mathcal{M}$ for which $A(\mathcal{M})$ is a subset of $\text{Max}(P)$ or $\text{Min}(P)$. Nevertheless, all posets have pure maximum chain matchings, as evidenced by the following Lemma, first exploited by Trotter in [26]. For the sake of completeness, the elementary argument for this fact is included here.

Lemma 5.1 *Every poset has a pure maximum chain matching.*

Proof Let P be a poset. Choose a maximum chain matching $\mathcal{M}$ in P which maximizes the quantity $q(\mathcal{M})$ defined by:

$$q(\mathcal{M}) = \sum_{a \in A(\mathcal{M})} |D(a)| \tag{2}$$

We claim that $\mathcal{M}$ is pure. To see this, suppose that there is a chain $C \in \mathbb{U}(\mathcal{M}) \cap \mathbb{D}(\mathcal{M})$. Then after a suitable relabeling of the chains in $\mathcal{M}$, there are elements $a, a' \in A(\mathcal{M})$ (not necessarily distinct), a positive integer r and a sequence $\{C_i = \{x_i < y_i\} : 1 \leq i \leq r\}$ of distinct chains from $\mathcal{M}$ so that $y_1 > a$ in P, $x_r < a'$ in P and $y_{i+1} > x_i$ in P whenever $1 \leq i < r$.

If $a \neq a'$, we could remove the r chains in the sequence from $\mathcal{M}$ and replace them by the following set of $r + 1$ chains:

$$\{a < y_1\}, \{x_r < a'\}, \{\{x_i < y_{i+1}\} : 1 \leq i < r\}$$

This would contradict the assumption that $\mathcal{M}$ is a maximum chain matching in P. We conclude that $a = a'$. In this case, we form a maximum chain matching $\mathcal{M}'$ from $\mathcal{M}$ by replacing the chains in the sequence by the following set:

$$\{x_r < a\}, \{\{x_i < y_{i+1}\} : 1 \leq i < r\}$$

Now the antichain $A(\mathcal{M}')$ is obtained from $A(\mathcal{M})$ by replacing a by y_1. Since $a < y_1$, we conclude that $q(M') \geq q(M) + 1$. The contradiction completes the proof of the lemma. □

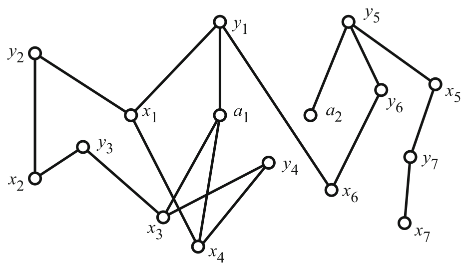

Fig. 7 Characterizing Chains in a Maximum Matching

5.2 The Proof of the Chain Matching Theorem

We are now ready to prove that a poset P with $\dim(P) = d \geq 4$ has a chain matching of size d. We argue by contradiction. Suppose this assertion is false and choose a counterexample P with $|P|$ minimum. Let u be any element of P. If $\dim(P - \{u\}) = \dim(P)$, then $P - \{u\}$ is also a counterexample, contradicting our choice of P. We conclude that $\dim(P - \{u\}) = \dim(P) - 1$. Since u was arbitrary, it follows that P is d-irreducible.

Let m be the maximum size of a chain matching in P. Clearly, $m = d - 1$; else P is not a minimum counterexample. Also, we note that P^*, the dual of P, is a counterexample of minimum size. This observation allows us to take advantage of duality in the arguments to follow.

Again, let $\mathcal{M}$ be any pure maximum chain matching in P. If $A(\mathcal{M}) \subseteq \mathrm{Max}(P)$, then Theorem 4.3 implies that $\dim(P) \leq m$. The contradiction shows that $A(\mathcal{M}) \not\subseteq \mathrm{Max}(P)$. By duality, we also know that $A(\mathcal{M}) \not\subseteq \mathrm{Min}(P)$. This implies that $\mathbb{U}(\mathcal{M}) \neq \emptyset \neq \mathbb{D}(\mathcal{M})$. Set

$$\mathbb{I}(\mathcal{M}) = \mathcal{M} - (\mathbb{D}(\mathcal{M}) \cup \mathbb{U}(\mathcal{M})).$$

Taking advantage of duality, we can assume that $|\mathbb{U}(\mathcal{M})| \leq |\mathbb{D}(\mathcal{M})|$. Since $m = d-1 \geq 3$, it follows that $m_0 = |\mathbb{D}(\mathcal{M}) \cup \mathbb{I}(\mathcal{M})| \geq 2$. Then let U be the subposet of P consisting of those points of P covered by chains in $\mathbb{U}(\mathcal{M})$. Let $D = P - U$. Since $\mathcal{M}$ is pure, it follows that U is an up set in P, D is a down set and both are non-empty. Furthermore, the maximum size of a chain matching in D is m_0. Also, the chains in $\mathbb{D}(\mathcal{M}) \cup \mathbb{I}(\mathcal{M})$ cover all points of D except the points in $A(\mathcal{M})$ which belong to $\mathrm{Max}(D)$. Then $\dim(D) \leq m_0$ by Theorem 4.3.

On the other hand, the $m - m_0$ chains in $\mathbb{U}(\mathcal{M})$ cover U, so $\mathrm{width}(U) \leq m - m_0$. Now Theorem 4.1 implies that $\dim(P) \leq \dim(D) + \mathrm{width}(U) \leq m_0 + (m - m_0) = m$. The contradiction completes the proof.

We comment further that if one seeks to prove that a 3-dimensional poset has a chain matching of size 3 without using the knowledge about the full list of 3-irreducible posets, then the only remaining detail is the case of a 3-irreducible poset P in which whenever $\mathcal{M}$ is a pure maximum chain matching in P, the sets $\mathbb{U}(\mathcal{M})$ and $\mathbb{D}(\mathcal{M})$ both consist of a single 2-element chain. This case requires only a few minutes work, but we elect to omit the details.

5.3 Matchings in Cover Graphs

It is tempting to believe that our chain matching theorem can be strengthened by requiring that the chains in the matching be covers. In this setting, a chain matching $\mathcal{M}$ can be viewed as a matching in the cover graph. However, we will now show that no such extension is possible.

Let $k \geq 2$, and set $d = \binom{2k}{k}$. We construct a height 3 poset P_k which contains the standard example S_d as a subposet, yet the largest matching in the cover graph of P_k has size $2k$. The minimal elements of P_k are labeled as $a(S)$ where S is a k-element subset of $\{1, 2, \ldots, 2k\}$. Similarly, the maximal elements of P_k are labeled as $b(T)$ where T is a k-element subset of $\{1, 2, \ldots, 2k\}$. There are $2k$ other elements of P_k and these are labeled as $m_1, m_2, \ldots, m_{2k}$. For each k-element subset S of $\{1, 2, \ldots, 2k\}$ and each integer i with $1 \leq i \leq 2k$, we have $a(S) < m_i$ and $m_i < b(S)$ if and only if $i \in S$. It follows that that $a(S) < b(T)$ unless S and T are complementary subsets of $\{1, 2, \ldots, 2k\}$. This shows that P_k contains the standard example S_d as a subposet. On the other hand, there are no covers between minimal and maximal elements, so a matching in the cover graph of P_k has size at most $|M| = 2k$.

Inverting parameters, it is clear that for large d, there is a poset P for which the dimension of P is d, yet the largest matching in the cover graph of P has size $O(\log d)$. The next theorem says that this bound is tight up to a multiplicative constant.

Theorem 5.2 *For every $m \geq 1$, if P is a poset and the maximum size of a matching in the cover graph of P is m, then the dimension of P is at most $(5^m + 2m)/2$.*

Proof We argue by contradiction. Suppose the inequality fails for some $m \geq 1$ and let P be a counterexample of minimum size. Then let $\mathcal{M}$ be a maximum matching in the cover graph of P. We label the edges in $\mathcal{M}$ as $\{C_i = \{x_i < y_i\} : 1 \leq i \leq m\}$. Now y_i covers x_i in P for all $i \in \{1, 2, \ldots, m\}$.

Let $X(\mathcal{M})$ denote the elements of P which are not covered by edges in $\mathcal{M}$. Unlike the case for matchings in the comparability graph, we no longer know that $X(\mathcal{M})$ is an antichain. Nevertheless, we know that for each $x \in X(\mathcal{M})$ any edges in the cover graph incident with x have their other end point in one of the covers in $\mathcal{M}$. It follows that for each $x \in X(\mathcal{M})$ and each $i \in \{1, 2, \ldots, m\}$, there are five possibilities:

(1) There are no edges in the cover graph with x as one endpoint and the other in C_i.
(2) x covers y_i.
(3) x covers x_i.
(4) y_i covers x.
(5) x_i covers x.

Accordingly, we may assign to each point $x \in X(\mathcal{M})$ a vector of length m with coordinate i being an integer from $\{1, 2, 3, 4, 5\}$ reflecting which of these possibilities holds for x and C_i.

Now suppose that x and y are distinct elements of $X(\mathcal{M})$ and they are both assigned to the same vector. Clearly, this implies that the subposet $Y = \{x, y\}$ is an autonomous subset in P. It follows that $\dim(P - \{x\}) = \dim(P)$. However, since $D(x) = D(y)$ and $U(x) = U(y)$, the cover graph of the subposet $P - \{x\}$ is obtained from the cover graph for P just by deleting x and the edges incident with x. It follows that $P - \{x\}$ is a counterexample of smaller size.

The contradiction shows that distinct points of $X(\mathcal{M})$ must be assigned to distinct vectors, so $|X(\mathcal{M})| \leq 5^m$. So altogether, P has at most $5^m + 2m$ points, and since $5^m + 2m \geq 7$, the conclusion of the theorem holds by appeal to Hiraguchi's inequality. □

6 Antichain Matchings

We are now ready to present the proof of our main theorem for antichain matchings by showing that if $\dim(P) = d \geq 3$, then P has an antichain matching of size d. Again, we proceed by contradiction and choose a counterexample P of minimum size. As in the preceding section, we use our knowledge concerning the list of 3-irreducible posets and note that P must therefore be a d-irreducible for some $d \geq 4$. Furthermore, the maximum size m of an antichain matching in P is $d - 1$.

Claim 1. There do not exist distinct minimal elements a and a' in P with $U(a) \subseteq U(a')$.

Proof If such a pair could be found in P, set $Q = P - \{a, a'\}$ and note that $\dim(Q) = d - 1 \geq 3$ by Theorem 3.7. This implies that there is an antichain matching $\mathcal{M}$ of size $d - 1$

in Q. It follows that $\{a, a'\} \cup \mathcal{M}$ is an antichain matching of size d in P. The contradiction completes the proof of Claim 1.

Claim 2. $d \geq 5$.

Proof Suppose to the contrary that $\dim(P) = 4$. Then $\mathrm{width}(P) \geq 4$. We consider a maximum antichain A in P. Suppose first that $A \subseteq \mathrm{Min}(P)$. Choose four distinct elements of A and label them as a_1, a_2, a_3 and a_4. By Claim 1, we may choose elements x_1 and x_2 from $U(A)$ so that $x_1 > a_1, x_2 > a_2, x_1 \parallel a_2$ and $x_2 \parallel a_1$.

Since $\{a_1, a_2\}$, $\{x_1, x_2\}$ and $\{a_3, a_4\}$ is an antichain matching of size 3, we know that $C = P - (A \cup \{x_1, x_2\})$ is a chain. But then $P - A$ can be covered by three chains and one of them (actually two of them) consists of a single point. It follows from Theorem 4.2 that $\dim(P) \leq 3$.

Using duality, we may assume that $U(A) \neq \emptyset \neq D(A)$. Since $|A| \geq 4$, it follows that at least one of $U(A)$ or $D(A)$ is a chain. Without loss of generality, we assume $D(A)$ is a chain. Let u_1 be the least element of $D(A)$. Then choose $a_1 \in A$ with $u_1 < a_1$. Since u_1 cannot be the only minimal element of P, there exists an element $a_2 \in A$ with $a_2 \parallel u_1$. Then a_2 is a minimal element in P, and by Claim 1, there is an element $x_2 \in U(A)$ with $x_2 > a_2$ and $x_2 \parallel u_1$. Since $\{x_2, u_1\}$, $\{a_1, a_2\}$ are antichains and $|A| \geq 4$, it follows that $U(A) - \{x_2\}$ has width 1. So $U(A)$ is covered by two chains, and one of them has size 1. Now Theorem 4.2 implies that $\dim(P - D(A)) \leq 2$. In turn, Theorem 4.1 implies that $\dim(P) \leq 3$. The contradiction completes the proof of Claim 2.

We now know that $\dim(P) \geq 5$. Choose distinct minimal elements a_1 and a_2 in P. By Claim 1, there are points x_1 and x_2 so that $x_1 > a_1, x_1 \parallel a_2, x_2 > a_2$ and $x_2 \parallel a_1$. Then $C_1 = \{a_1 < x_1\}$ and $C_2 = \{a_2 < x_2\}$ are disjoint incomparable chains. Set $Q = P - (C_1 \cup C_2)$. Then Lemma 3.9 implies that $\dim(Q) \geq d - 2 \geq 3$, so Q has an antichain matching $\mathcal{M}$ of size $d - 2$. It follows that $\mathcal{M} \cup \{\{a_1, a_2\}, \{x_1, x_2\}\}$ is an antichain matching of size d in P. The contradiction completes the proof of our antichain matching theorem.

7 Open Problems

First, it would be of interest to characterize posets for which the dimension is exactly equal to the size of a maximum chain (or antichain) matching.

Second, we see some hope of extending the ideas in this paper to finding a compact proof of a "true" theorem for which no complete proof has ever been published:

Theorem 7.1 *Let $d \geq 4$, and let P be a poset with $|P| \leq 2d + 1$. Then $\dim(P) < d$ unless P contains the standard example S_d.*

As our bounds regarding matchings in the cover graph are relatively tight, it may be possible to determine for each $m \geq 1$, the maximum dimension d_m of a poset P where the maximum size of a matching in the cover graph of P has size m. It may even be the case that the example presented is extremal.

One of the referees raised the following question. Is it true that if P is a poset and $\dim(P) = d \geq 3$, then there is a matching of size d in the incomparability graph of P in which each pair in the matching is a critical pair in P? We were aware of this question, but recognized that the techniques developed here will not settle this question, at least without significant modification. The difficulty we see is that the concept of being a critical pair becomes weaker in subposets, i.e, when Q is a subposet of P, a critical pair in Q will be an incomparable pair in P although not necessarily a critical pair. Still our intuition is that the referee's question has an affirmative answer.

Finally, there may be some insights here which could be applied to the Removable Pair Conjecture, which has proven to be a considerable challenge for more than 60 years, and that was our real goal in working on these topics.

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