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Posets and VPG Graphs

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Abstract We investigate the class of intersection graphs of paths on a grid (VPG graphs), and specifically the relationship between the bending number of a cocomparability graph and the poset dimension of its complement. We show that the bending number of a cocomparability graph G is at most the poset dimension of the complement of G minus one. Then, via Ramsey type arguments, we show our upper bound is best possible.

Keywords Poset · Dimension · Bending number · Cocomparability graph · Product Ramsey theorem

1 Background

Let $\mathcal{R}$ be a set of simple paths on a grid. The *intersection graph of paths on a grid* (VPG graph) has a set of vertices corresponding to the paths of $\mathcal{R}$ and two vertices are adjacent if and only if the corresponding paths intersect in at least one grid point. A *string graph* is the intersection graph of curves in the plane, defined similarly to a VPG graph, but on a set of simple paths in the plane. The two classes, VPG graphs and string graphs, are known to be equivalent. (The reader is referred to [2] for further discussion and results on VPG graphs.)

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For a representation $\mathcal{R}$ of a VPG graph G, we may consider the maximum number of bends (changes of direction) associated with any of its paths. Given a VPG graph G, define the *bending number* of G, denoted by $b(G)$, as the minimum value taken over all VPG representations of G. There are VPG graphs with arbitrarily high bending number [12].

Let P be a partially ordered set on a ground set X. A *realizer* of P is a set of linear extensions of P, such that their intersection equals P. The *poset dimension* is the size of a realizer of minimum cardinality over all realizers of P. We denote the poset dimension of a poset P by $dim(P)$. A *comparability graph* is defined over a poset P, where each vertex corresponds to an element in X and two vertices u, v are adjacent if their corresponding elements are comparable. The complement of a comparability graph is called a *cocomparability graph*. It is known that the poset dimension is a *comparability invariant*, meaning that all transitive orientations of a given comparability graph have the same dimension. Thus, we may extend the definition of the poset dimension to the comparability graph H of P and denote it by $dim(H)$. For further reading on the subject the reader is referred to [14].

It was shown in Golumbic et al. [10] that all cocomparability graphs are string graphs,[1] with representations known as function diagrams, and therefore they are also VPG graphs. In their paper, they showed further that a cocomparability graph has representation of concatenated permutation diagrams (see Fig. 1), obtained from a realizer of the poset, and gave the following bound on its size in terms of the poset dimension.

Theorem 1.1. *Golumbic et al.* [10] *If G is a cocomparability graph, then the minimum number of permutation diagrams in a permutation concatenation representation equals* $dim(\overline{G}) - 1$*, the poset dimension of the complement of the cocomparability graph minus one.*

Fox and Pach [8] have recently studied cocomparability graphs as string graphs from the point of view of extremal graph theory showing that most string graphs contain huge subgraphs that are cocomparability graphs. Specifically, for every $\epsilon > 0$ there exists $\delta > 0$ with the property that if $\mathcal{C}$ is a collection of curves whose string graph has at least $\epsilon|\mathcal{C}|^2$ edges, then one can select a subcurve γ' of each $\gamma \in \mathcal{C}$ such that the string graph of the collection $\{\gamma' \mid \gamma \in \mathcal{C}\}$ has at least $\delta|\mathcal{C}|^2$ edges and is a cocomparability graph.

The poset dimension has relations to other graph parameters, such as the boxicity of a graph, see [1]. We show here a relationship between the bending number of a cocomparability graph and the poset dimension of its complement.

2 VPG-Bending Number and Poset Dimension

The following theorem shows that the bending number of a cocomparability graph G is bounded by the poset dimension of its complement $\overline{G}$. The proof is constructive as we take a permutation concatenation representation of G and transform it into a VPG representation, such that for every permutation diagram in the representation, we use exactly one bend. See Figs. 1 and 2 for an example of the construction.

[1]The complement of a comparability graph is also known as an incomparability graph.

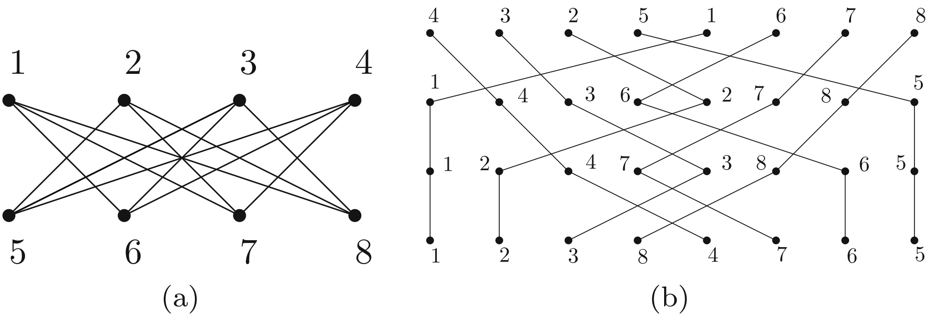

Fig. 1 The Hiraguchi graph on 8 vertices and a concatenation of 3 permutation diagrams representing its complement

Theorem 2.1. *Let G be a cocomparability graph, then $b(G) \leq dim(\overline{G}) - 1$.*

Proof Let G be a cocomparability graph with $dim(\overline{G}) = k$ and let $\mathcal{R}$ be a minimum representation of G as $k-1$ concatenated permutation diagrams. Denote the set of k consecutive linear orders of $\mathcal{R}$ by $\{\alpha_1, \ldots, \alpha_k\}$ and the members of α_i by $\{\alpha_{i,1}, \ldots, \alpha_{i,n}\}$, for $i = 1, \ldots, k$ and $n = |V(G)|$. Consider an (x, y) axis system and label the coordinate (x, y) by $\alpha_{i,j}$, for $1 \leq i \leq k$ and $1 \leq j \leq n$ according to the following conditions:

(1.) if $i \pmod 4 \equiv 1$, then $x = 0$ and $y = \lfloor \frac{i}{4} \rfloor \times n + j$.

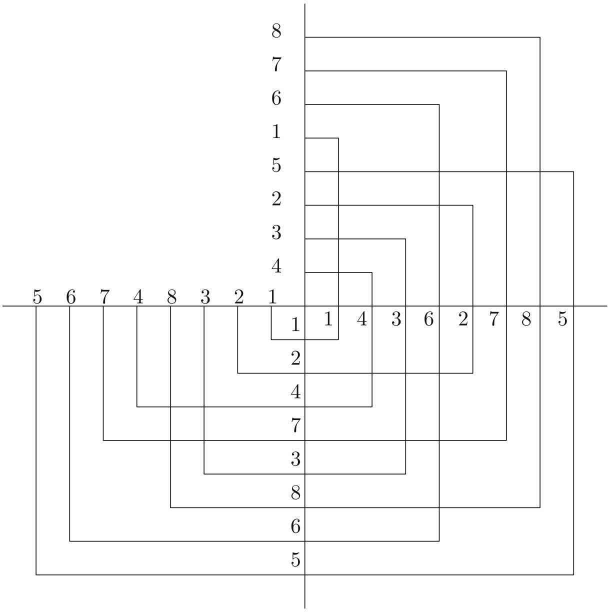

Fig. 2 A 3-bend VPG representation for the complement of the Hiraguchi graph in Fig. 1

(2.) if i (mod 4) $\equiv 2$, then $x = \lfloor \frac{i}{4} \rfloor \times n + j$ and $y = 0$.
(3.) if i (mod 4) $\equiv 3$, then $x = 0$ and $y = -(\lfloor \frac{i}{4} \rfloor \times n + j)$.
(4.) if i (mod 4) $\equiv 0$, then $x = -(\lfloor \frac{i}{4} \rfloor \times n + j)$ and $y = 0$.

Denote by $\alpha_{i,j}^{-1}$ the (x, y) coordinates $\alpha_{i,j}$ is assigned to, for $1 \leq i \leq k$ and $1 \leq j \leq n$.

Let $\{\alpha_{1,j_1}, \ldots, \alpha_{k,j_k}\}$ be the set of occurrences of a vertex v in $\mathcal{R}$. We create a path for v with $k - 1$ bends. We start the path from α_{1,j_1}^{-1} and continue it until it reaches α_{2,j_2}^{-1}, then we bend it 90° to the right. We continue the path in a clockwise outgoing spiral fashion, bending it at each α_{l,j_l}^{-1}, for $l = 3, \ldots, k$. We repeat this construction for every vertex in G to obtain a VPG representation of G with $k - 1$ bends per path. Notice that two paths in the above construction intersect if and only if the corresponding line segments in $\mathcal{R}$ intersect, as each permutation diagram in $\mathcal{R}$ is represented in a distinct area in the constructed VPG representation. □

Remark Alternative constructions can also be obtained for any prescribed shape, for example, by alternating bendings to the right and to the left, we get a staircase instead of an outgoing spiral.

The relationship between the parameters $b(G)$ and $dim(\overline{G})$ in Theorem 2.1 can vary widely. On the one hand, the bending number of an interval graph equals 0, yet the poset dimension of its complement (i.e., the poset dimension of an interval order) can be arbitrarily high [14]. But, on the other hand, a non-trivial permutation graph G satisfies $dim(G) = dim(\overline{G}) = 2$, so its bending number must be either 0 or 1.

The chordless 4-cycle C_4 is a permutation graph whose bending number is 0, so there is a gap of one in the inequality for this example. The 4-wheel W_4 is a permutation graph whose bending number is 1, so the inequality is tight for this example. Cohen, Golumbic and Ries [5] have answered this question for the case of cographs:

Theorem 2.2. *If G is a cograph graph, then $b(G) = 0$ if and only if G contains no induced W_4.*

Finding a characterization of permutation graphs with $b(G) = 0$ is posed as an open question in Section 4. In the following lemma, we show a connection between bipartite permutation graphs and B_0-VPG graphs.

A graph is a *bipartite permutation graph*, if it is both bipartite and a permutation graph. Let $G = (A \cup B, E)$ be a bipartite graph where A and B are independent sets. An ordering $<$ of A has the *adjacency property*, if for every vertex $b \in B$, $N(b)$ consists of vertices that are consecutive in $<$. The family of *convex graphs* is the set of all bipartite graphs that satisfy the *adjacency property*. The following class relations hold:

Lemma 2.3. *Bipartite permutation $\subsetneq$ convex $\subsetneq$ bipartite B_0-VPG.*

Proof The relation bipartite permutation $\subsetneq$ convex is known from [13].

Consider a convex graph and an ordering $<$ of A satisfying the adjacency property. Create vertical parallel segments for the set A according to the ordering $<$ and a horizontal segment for each vertex in B, such that it intersects the vertical segments according to their order in $<$, thus obtaining a B_0-VPG representation for the graph. Clearly, the graph C_6 is not a convex graph, however, it has a simple B_0-VPG representation. Hence, we have the relation convex $\subsetneq$ bipartite B_0-VPG. □

Corollary 2.4. *Bipartite permutation graphs and convex graphs have VPG bending number 0.*

Bipartite B_0-VPG graphs are known to be equivalent to the Grid Instersection Graphs (GIG) of [3], see [2]. Other results on graphs with bending number 0 appear in [4, 9]. Besides interval graphs, there are other cocomparability graphs with small bending number and high poset dimension. For example, the following result shows that the complement of the Hiraguchi graph on $2n$ vertices (also know as the Standard Example or S_n in the book [14]) has small bending number, while it is known that the poset dimension of S_n equals n.

Proposition 2.5. *The complement of the Hiraguchi graph has bending number one.*

Proof An example of a single bend representation for the complement of the Hiraguchi graph with 8 vertices is given in Fig. 3. It is easy to see that this representation can be generalized to handle the complement of the Hiraguchi graph with n vertices. □

In the next section, we show that our upper bound is best possible, that is, there exists a cocomparability graph G satisfying $k = b(G) = dim(\overline{G}) - 1$, for each value $k \geq 2$.

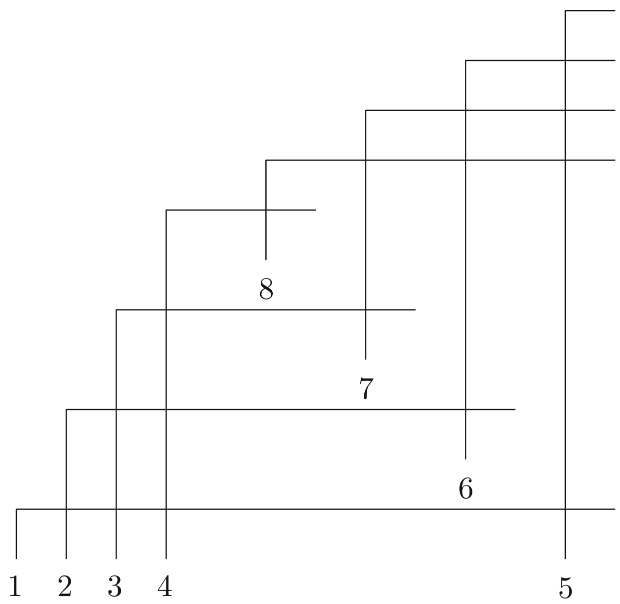

Fig. 3 A single bend VPG representation for the complement of the Hiraguchi graph in Fig. 1

3 Cocomparability Graphs of High Poset Dimension and High Bending Number

Let $\mathbf{n}$ denote the chain $1 < 2 < \cdots < n$, and let $\mathbf{n}^t$ denote the cartesian product of t copies of $\mathbf{n}$, so that $(i_1, i_2, \ldots, i_t) \leq (j_1, j_2, \ldots, j_t)$ in $\mathbf{n}^t$ if and only if $i_k \leq j_k$ in $\mathbf{n}$ for $k = 1, 2, \ldots, t$. Let G denote the cocomparability graph of $\mathbf{n}^t$. It is well known that $\mathbf{n}^t$ has dimension t when $n \geq 2$.

The goal of this section is to prove the following theorem.

Theorem 3.1. *For every integer $t \geq 1$, there exists an integer n_0 so that if $n \geq n_0$, then $b(G) = t - 1$, where G is the cocomparability graph of $\mathbf{n}^t$.*

For each $a \in \mathbf{n}^t$, let $P(a)$ denote the path in a VPG representation $\mathcal{R}$ that corresponds to a. Theorem 3.1 holds trivially for $t = 1$. For $t = 2$, let n_0 be at least 9. Then we argue by contradiction and assume there exists a 0-bend VPG representation of $\mathbf{n}^t$. Note that each path $P(a)$ is either a horizontal segment, or a vertical segment. By the pigeonhole principle, there exist three horizontal or vertical segments in $\{P(1,9), P(3,7), P(5,5), P(7,3), P(9,1)\}$. Without loss of generality, we may assume $P(1,9)$, $P(3,7)$ and $P(5,5)$ are horizontal segments. Since all three segments overlap, there exist two segments from $\{P(1,9), P(3,7), P(5,5)\}$ such that their intersection is contained in the third one. However, $P(1,8)$ intersects both $P(3,7)$ and $P(5,5)$, but not $P(1,9)$; $P(6,6)$ intersects both $P(3,7)$ and $P(1,9)$, but not $P(5,5)$; $P(2,6)$ intersects both $P(1,9)$ and $P(5,5)$, but not $P(3,7)$. The contradiction shows that for $t = 2$, $b(G) = 1$ when $n \geq 9$.

From now on, we may assume $t \geq 3$. In the context below, $n_0, n_1, n_2, \ldots, n_9$ are integers where $n_0 \gg n_1 \gg n_2 \gg n_3 > n_4 \gg \cdots \gg n_9 \gg t$.

The basic idea of the proof is straightforward. We will assume that we have a $(t-2)$-bend VPG representation of $\mathbf{n}^t$ and argue to a contradiction – provided n is sufficiently large. The issue as to how large n must be in order to reach this contradiction will depend on t.

Now suppose $b(G) \leq t - 2$. Therefore, there exists a VPG representation $\mathcal{R}$ of $\mathbf{n}^t$ with at most $t-2$ bends. For each $a \in \mathbf{n}^t$, let $a[l]$ denote the l-th coordinate of a. For each $P(a)$, we arbitrarily choose one end as its starting point. Since $P(a)$ has at most $t-2$ bends, let $S_1(a), S_2(a), \ldots, S_{t-1}(a)$ denote the $t-1$ consecutive segments of $P(a)$. (It is easy to see that we may assume each $P(a)$ has exactly $t-1$ segments.) Since all paths in $\mathcal{R}$ are paths in a rectangular grid, each segment $S_k(a)$ is either vertical or horizontal. Let $V_k(a)$ denote the y-coordinate (x-coordinate, resp.) of segment $S_k(a)$ if $S_k(a)$ is horizontal (vertical, resp.).

3.1 Product Ramsey Theorem

The following Product Ramsey theorem will be used in making certain uniformizing assumptions about the VPG representation. The reader is referred to [11] for further reading on this subject.

Theorem 3.2. *Felsner et al.* [6] *Given positive integers m, u, r and t, there exists an integer n_p so that if $n \geq n_p$ and f is any map which assigns to each $\mathbf{u}^t$ grid of $\mathbf{n}^t$ a color from $\{1, 2, \ldots, r\}$, then there exists a subposet P isomorphic to $\mathbf{m}^t$ and a color $\alpha \in \{1, 2, \ldots, r\}$ so that $f(g) = \alpha$ for every $\mathbf{u}^t$ grid g from P.*

Define the Product Ramsey number $PR(m, u, r, t)$ be the least n_p for which the conclusion of the preceding theorem holds. In Theorem 3.2, the resulting subposet P is isomorphic to $\mathbf{m}^t$. Hence, whenever there is a VPG representation of P, the same representation is also a VPG representation of $\mathbf{m}^t$. So in the context below, we will not distinguish P from $\mathbf{m}^t$.

For each $a \in \mathbf{n}^t$, we consider a vector, denoted $val(a)$, with $m(t) = t-1+\frac{1}{2}(t-1)(t-2)$ coordinates, defined as follows. The first $t-1$ coordinates of $val(a)$ are chosen from $\{\rightarrow, \leftarrow, \uparrow, \downarrow\}$, according to the direction of $S_1(a), S_2(a), \ldots, S_{t-1}(a)$. The last $\frac{1}{2}(t-1)(t-2)$ coordinates of $val(a)$ correspond to ordered pairs $(S_k(a), S_l(a))$ where $1 \le k < l \le t-1$. A coordinate for $(S_{k_1}(a), S_{l_1}(a))$ occurs before a coordinate for $(S_{k_2}(a), S_{l_2}(a))$ when (1) $k_1 < k_2$ or (2) $k_1 = k_2$ and $l_1 < l_2$, i.e., the last $\frac{1}{2}(t-1)(t-2)$ coordinates of $val(a)$ are ordered lexicographically. Moreover, the coordinate for $(S_k(a), S_l(a))$ is then chosen from $\{+, 0, -\}$, according to the sign of $V_k(a) - V_l(a)$.

In total there are less than $4^{m(t)}$ possible different vectors. By Theorem 3.2, when $n_0 > PR(n_1, 1, 4^{m(t)}, t)$, there exists a subposet P_1 of $\mathbf{n}^t$ isomorphic to $\mathbf{n}_1^t$, so that for any $a, b \in \mathbf{n}_1^t$, $val(a) = val(b)$. Without loss of generality, we may assume for every $a \in \mathbf{n}_1^t$, the first coordinate of $val(a)$ is "$\rightarrow$", i.e., $S_1(a)$ is horizontal and goes from left to right.

Note that if $val(a) = val(b)$ for any $a, b \in \mathbf{n}_1^t$, then any two paths in the VPG representation of $\mathbf{n}_1^t$ looks alike and roughly speaking they have the same "shape".

Now we will make some uniformizing assumptions for the first segments of all paths in the VPG representation of $\mathbf{n}_1^t$. Fix a subset $S \in [t]$, consider any $\mathbf{2}^t$ grid $g = \{c_1, d_1\} \times \{c_2, d_2\} \times \cdots \times \{c_t, d_t\}$ of $\mathbf{n}_1^t$ with $c_k < d_k$ for each $1 \le k \le t$. Let $a[k] = c_k, b[k] = d_k$ if $k \in S$; $a[k] = d_k, b[k] = c_k$ if $k \in [t] \backslash S$. We assign color 0 to such $\mathbf{2}^t$ grid g if $V_1(a) = V_1(b)$; color 1 if $V_1(a) \neq V_1(b)$. By Theorem 3.2, when $n_1 > PR(n_2, 2, 2, t)$, there exists a subposet P_2 of $\mathbf{n}_1^t$ isomorphic to $\mathbf{n}_2^t$ and a color $\alpha_S \in \{0, 1\}$, so that every $\mathbf{2}^t$ grid from P_2 is colored α_S.

Then repeat the above process for every subset $S \in [t]$. (There are 2^t such subsets.) Let P_3 be the resulting poset which is isomorphic to $\mathbf{n}_3^t$.

3.2 All Collinear Segments and Strictly Parallel Segments

For every $1 \le k \le t-1$, we say the k-th segments are *all collinear* if for any $b_1, b_2 \in \mathbf{n}^t$ (or a subposet of $\mathbf{n}^t$), $V_k(b_1) = V_k(b_2)$; the k-th segments are *strictly parallel* if for any distinct $b_1, b_2 \in \mathbf{n}^t$ (or a subposet of $\mathbf{n}^t$), $V_k(b_1) \neq V_k(b_2)$.

Suppose there exists an $S \in [t]$ such that $\alpha_S = 0$. Let the element $a \in \mathbf{n}_3^t$ be defined by $a[l] = 1$ if $l \in S$, $a[l] = n_3$ if $l \in [t] \backslash S$. Let $\mathbf{n}_4^t = \{b \in \mathbf{n}_3^t : b[l] \neq 1, b[l] \neq n_3, 1 \le l \le t\}$. Then for any $b_1, b_2 \in \mathbf{n}_4^t$, $V_1(b_1) = V_1(a) = V_1(b_2)$. Hence the first segments of $\mathbf{n}_4^t$ are all collinear.

Otherwise, $\alpha_S = 1$ for all $S \in [t]$. Note that when $n_4 > n_5^t$, there exists a subposet P_5 of $\mathbf{n}_4^t$ isomorphic to $\mathbf{n}_5^t$ so that for any distinct $b_1, b_2 \in P_5$, $b_1[l] \neq b_2[l]$ for $l = 1, 2, \ldots, t$. Hence $V_1(b_1) \neq V_1(b_2)$ for any $b_1, b_2 \in \mathbf{n}_5^t$. i.e., the first segments of $\mathbf{n}_5^t$ are strictlyparallel.

Therefore, the first segments of $\mathbf{n}_5^t$ are either all collinear, or strictly parallel. We can make similar uniformizing assumptions for every segments in the VPG representation of $\mathbf{n}_5^t$ by repeating the above process. Let P_6 be the resulting poset which is isomorphic to $\mathbf{n}_6^t$. Then, for every $1 \le k \le t-1$, the k-th segments of $\mathbf{n}_6^t$ are either all collinear, or strictly parallel. Let T_1 denote the set of all k where the k-th segments of $\mathbf{n}_6^t$ are all collinear. Similarly, Let T_2 denote the set of all k where the k-th segments of $\mathbf{n}_6^t$ are strictly parallel.

Let P be a poset and let f be a function maps P to $\mathbb{R}$. The following definitions are from Felsner et al. [6]. We say f is *monotonic* if it is either order-preserving or order-reversing. Now consider an order-preserving function f which maps $\mathbf{n}^t$ (or a subposet of $\mathbf{n}^t$) to $\mathbb{R}$. We say that f is *dominated* by coordinate α if for all x and y from its domain, $f(x) < f(y)$ whenever $x(\alpha) < y(\alpha)$. Dually, given an order-reversing function f, we say that f is *dominated* by coordinate α if for all x and y from its domain, $f(x) > f(y)$ whenever $x(\alpha) < y(\alpha)$.

Theorem 3.3. *Fishburn et al.* [7] *Given integers m and t, there exists an integer n_q so that if $n \geq n_q$ and f is any injective function from $\mathbf{n}^t$ to $\mathbb{R}$, then there exists a coordinate $\beta \in \{1, 2, \ldots, t\}$ and a subposet P isomorphic to $\mathbf{m}^t$ so that the restriction of f to P is monotonic and dominated by coordinate β.*

We will refer to the least n_q for which the conclusion of the preceding theorem holds as the Dominating Coordinate number $DC(m, t)$. Similar to Theorem 3.2, the resulting subposet P in Theorem 3.3 is isomorphic to $\mathbf{m}^t$. Hence, whenever there is a VPG representation of P, the same representation is also a VPG representation of $\mathbf{m}^t$. So in the context below, we will not distinguish P from $\mathbf{m}^t$.

Let $k \in T_2$. Define an injective function $f_k : \mathbf{n}_6^t \to \mathbb{R}; a \mapsto V_k(a)$ for every $a \in \mathbf{n}_6^t$. By Theorem 3.3, when $n_6 > DC(n_7, t)$, there exists a coordinate $\beta_k \in \{1, 2, \ldots, t\}$ and a subposet P_7 isomorphic to $\mathbf{n}_7^t$ so that the restriction of f_k to P_7 is monotonic and dominated by coordinate β_k.

Repeat the above process for every $k \in T_2$. (There are at most $t - 1$ of such k.) Let P_8 be the resulting poset which is isomorphic to $\mathbf{n}_8^t$. By Theorem 3.3, each f_k, $k \in T_2$, is dominated by some coordinate $\beta_k \in \{1, 2, \ldots, t\}$. Note that $|T_2| \leq t - 1$. Without loss of generality, we may assume every f_k, $k \in T_2$, is dominated by one of the first $t - 1$ coordinates in $\mathbf{n}_8^t$, i.e., $\{\beta_k : k \in T_2\} \subseteq \{1, 2, ..., t - 1\}$. Note that it is possible that two functions f_k and f_l, $k, l \in T_2, k \neq l$, are dominated by a same coordinate.

Given any $\mathbf{2}^t$ grid $g = \{c_1, d_1\} \times \{c_2, d_2\} \times \cdots \times \{c_t, d_t\}$ of $\mathbf{n}_8^t$ with $c_i < d_i$ for each $1 \leq i \leq t$, let $a = (c_1, c_2, ..., c_{t-1}, d_t)$, $b = (d_1, d_2, ..., d_{t-1}, c_t)$. Note that a and b are incomparable in $\mathbf{n}_8^t$. Therefore $P(a)$ and $P(b)$ intersect at one or more grid points. Define a map f: $\mathbf{2}^t \to \mathbb{N}_+^2$; $g \mapsto (k, l)$ for every $\mathbf{2}^t$ grid $g \in \mathbf{n}_8^t$, where (k, l) is the smallest (in lexicographic order) ordered pair such that $S_k(a)$ and $S_l(b)$ intersect. Clearly, the total number of possible (k, l) pairs is at most $(t - 1)^2$. By Theorem 3.2, when $n_8 > PR(n_9, 2, (t - 1)^2, t)$, there exists a subposet P_9 isomorphic to $\mathbf{n}_9^t$ and a color $\gamma = (i, j)$, $1 \leq i, j \leq t - 1$ so that $f(g) = \gamma$ for every $\mathbf{2}^t$ grid g from P_8.

Claim 3.4. *All the i-th segments are orthogonal to all the j-th segments in $\mathbf{n}_9^t$.*

Proof As all the i-th segments in $\mathbf{n}_9^t$ are parallel, as for all the j-th segments, suppose for a contradiction that they are parallel to all the i-th segments. Consider $a_1, a_2, b_1, b_2 \in \mathbf{n}_9^t$, where $a_1 = (1, 1, ..., 1, 3)$, $a_2 = (2, 2, ..., 2, 4)$, $b_1 = (3, 3, ..., 3, 1)$, $b_2 = (4, 4, ..., 4, 2)$. Note that $S_j(b_1)$ and $S_j(b_2)$ are parallel to both $S_i(a_1)$ and $S_i(a_2)$ by our assumption. Also $S_j(b_1)$ and $S_j(b_2)$ intersect both $S_i(a_1)$ and $S_i(a_2)$. Hence, $S_j(b_1)$ and $S_j(b_2)$ overlap both $S_i(a_1)$ and $S_i(a_2)$. Then either $S_j(b_1)$ overlaps $S_j(b_2)$, or $S_i(a_1)$ overlaps $S_i(a_2)$. Both cases contradict the fact that $a_1 < a_2$ and $b_1 < b_2$ in $\mathbf{n}_9^t$. □

Claim 3.5. *The i-th and j-th segments are both strictly parallel in $\mathbf{n}_9^t$.*

Proof Suppose the i-th segments are all collinear. Consider $a_1, a_2, b \in \mathbf{n}_9^t$, where $a_1 = (1, 1, ..., 1, 2)$, $a_2 = (2, 2, ..., 2, 3)$, $b = (3, 3, ..., 3, 1)$. By Claim 3.4, $S_j(b)$ is orthogonal to both $S_i(a_1)$ and $S_i(a_2)$. Note that $S_j(b)$ intersects both $S_i(a_1)$ and $S_i(a_2)$. Since the i-th segments are all collinear, $V_i(a_1) = V_i(a_2)$. Hence, $S_i(a_1)$ overlaps $S_i(a_2)$. This contradicts the fact that $a_1 < a_2$ in $\mathbf{n}_9^t$. Similarly, the j-th segments are also strictly parallel. □

For any $a, b, c \in \mathbf{n}_9^t$ and $1 \leq k \leq t-1$, we say $S_k(b)$ is *trapped by* $S_k(a)$ and $S_k(c)$ if either $V_k(a) \geq V_k(b) \geq V_k(c)$ or $V_k(a) \leq V_k(b) \leq V_k(c)$.

Claim 3.6. *For any $a, b, c \in \mathbf{n}_9^t$, $S_k(b)$ is trapped by $S_k(a)$ and $S_k(c)$ for all $1 \leq k \leq t-1$ if $a[l] > b[l] > c[l]$ for all $1 \leq l \leq t-1$.*

Proof The k-th segments of $\mathbf{n}_9^t$ are either all collinear, or strictly parallel. If the k-th segments are all collinear, then $V_k(a) = V_k(b) = V_k(c)$. If the k-th segments are strictly parallel, then f_k is dominated by a coordinate $\beta_k \in \{1, 2, ..., t-1\}$. We have either $V_k(a) > V_k(b) > V_k(c)$ (when f_k is order preserving) or $V_k(a) < V_k(b) < V_k(c)$ (when f_k is order reversing). □

Claim 3.7. $j \in \{1, t-1\}$.

Proof As $1 \leq j \leq t-1$, the claim holds trivially for $t = 3$. For $t \geq 4$, suppose $j \notin \{1, t-1\}$, then $j+1, j-1 \in \{1, 2, ..., t-1\}$. Consider $b_1, b_2, b_3, a \in \mathbf{n}_9^t$, where $b_1 = (4, 4, ..., 4, 1)$, $b_2 = (3, 3, ..., 3, 4)$, $b_3 = (2, 2, ..., 2, 2)$, $a = (1, 1, ..., 1, 3)$. Since $f(g) = (i, j)$ for every $\mathbf{2}^t$ grid g in $\mathbf{n}_9^t$, $S_i(a)$ intersects both $S_j(b_1)$ and $S_j(b_3)$, but not $P(b_2)$. Note that $b_1[k] > b_2[k] > b_3[k]$ for $k = 1, 2, ..., t-1$. Hence, $S_l(b_2)$ is trapped by $S_l(b_1)$ and $S_l(b_3)$ for $l = j-1, j, j+1$. Since $S_i(a)$ intersects both $S_j(b_1)$ and $S_j(b_3)$, we must have $S_i(a)$ intersects $S_j(b_2)$. This contradicts the fact that $a < b_2$ in $\mathbf{n}_9^t$. □

Similarly, we must have $i \in \{1, t-1\}$. Note that since all the i-th segments are orthogonal to all the j-th segments, i cannot be equal to j. So either $i = 1, j = t-1$ or $i = t-1, j = 1$. Without loss of generality, we may assume the former is true.

Consider $c_1, c_2, c_3, v, w, z \in \mathbf{n}_9^t$, where $c_1 = (5, 5, ..., 5, 1)$, $c_2 = (4, 4, ..., 4, 4)$, $c_3 = (3, 3, ..., 3, 5)$, $v = (6, 6, ..., 6, 3)$, $w = (2, 2, ..., 2, 2)$, $z = (1, 1, ..., 1, 6)$ (See Fig. 4). By Claim 3.5 and Claim 3.7, we have $\{1, t-1\} \subseteq T_2$. Hence both injective functions f_1 and f_{t-1} are monotonic and dominated by one of the first $t-1$ coordinates in $\mathbf{n}_9^t$. We assume

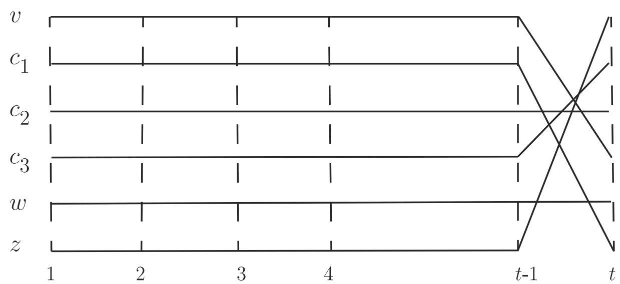

Fig. 4 A subposet of $\mathbf{n}_9^t$

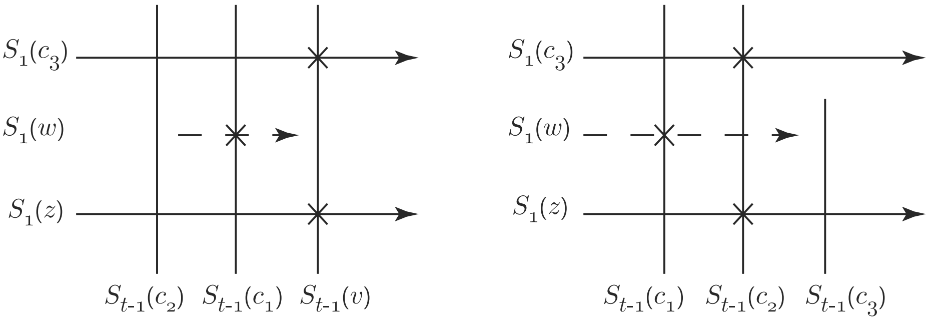

Fig. 5 A partial VPG representation of a subposet of $\mathbf{n}_9^t$

with out loss of generality that f_1 is order preserving, and recall that $S_1(a)$ is horizontal and goes from left to right for all $a \in \mathbf{n}_9^t$.

Case 1 f_{t-1} is order preserving (See Fig. 5).

f_{t-1} is order preserving implies $S_{t-1}(c_1)$ is on the left of $S_{t-1}(v)$. Since v is incomparable to both z and c_3, $S_{t-1}(v)$ intersects $S_1(c_3)$ and $S_1(z)$. Similarly, $S_{t-1}(c_1)$ intersects $S_1(w)$. By Claim 3.6, segment $S_k(w)$ is trapped by $S_k(c_3)$ and $S_k(z)$ for $k = 1, 2$. Hence, we must have $S_{t-1}(v)$ intersects $S_1(w)$. This contradicts the fact that $w < v$ in $\mathbf{n}_9^t$.

Case 2 f_{t-1} is order reversing (See also Fig. 5).

f_{t-1} is order reversing implies $S_{t-1}(c_1)$ is on the left of $S_{t-1}(c_2)$. Since c_2 is incomparable to both c_3 and z, $S_{t-1}(c_2)$ intersects $S_1(c_3)$ and $S_1(z)$. Similarly, $S_1(w)$ intersects $S_{t-1}(c_1)$. By Claim 3.6, segment $S_k(w)$ is trapped by $S_k(c_3)$ and $S_k(z)$ for $k = 1, 2$. Hence, we must have $S_{t-1}(c_2)$ intersects $S_1(w)$. This contradicts the fact that $w < c_2$ in $\mathbf{n}_9^t$. The contradiction completes the proof of Theorem 3.1.

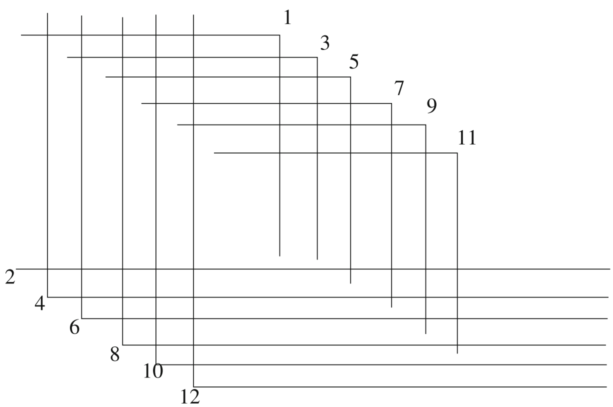

Fig. 6 A single bend VPG representation for the complement of the chordless path $\overline{P_{12}}$

4 Open Questions and Further Comments

We propose further study of the relationship between the parameters $b(G)$ and $dim(\overline{G})$ for cocomparability graphs. Among the questions that could be investigated are the following:

(1) Find additional examples with low bending number and high poset dimension.
(2) Characterize the permutation graphs for which the VPG bending number equals to 0?

Answering a question we had posed earlier, an anonymous referee has pointed out that the bending number $b(\overline{C_n})$ of the complement of a chordless cycle is 1, for $n \geq 6$, as follows: Begin with a representation of the complement of the chordless path $\overline{P_n}$, see Fig. 6 for an example. To obtain $\overline{C_n}$: If n is even, shorten the vertical segment of vertex n, so that it does not intersect vertex 1. If n is odd, shorten the vertical segment of vertex 1, so that it does not intersect vertex n.

The referee also points out that this construction can be generalized to the complement of any convex graph: Define "circular convex" by replacing the order in the definition of convex by a circular order. Then the construction for complements of even cycles generalizes to complements of circular convex graphs. So we obtain another representation of Hiraguchi graphs (which are circular convex) with 1 bend.

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