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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 61 (2017) 1047–1053 www.elsevier.com/locate/endm

Boolean Dimension and Local Dimension

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Abstract

Dimension is a standard and well-studied measure of complexity of posets. Recent research has provided many new upper bounds on the dimension for various structurally restricted classes of posets. Bounded dimension gives a succinct representation of the poset, admitting constant response time for queries of the form "is $x < y$?". This application motivates looking for stronger notions of dimension, possibly leading to succinct representations for more general classes of posets. We focus on two: *boolean dimension*, introduced in the 1980s and revisited in recent research, and *local dimension*, a very new one. We determine precisely which values of dimension/boolean dimension/local dimension imply that the two other parameters are bounded. This is an extended abstract; see arXiv:1705.09167 for a full version.

Keywords: Partially ordered sets, order dimension

http://dx.doi.org/10.1016/j.endm.2017.07.071
1571-0653/

1 Introduction

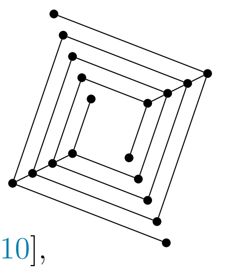
Dimension

The *dimension* of a poset $P = (X, \leqslant)$ is the minimum number of linear extensions of $\leqslant$ on X whose intersection is $\leqslant$. More precisely, a *realizer* of a poset $P = (X, \leqslant)$ is a set $\{\leqslant_1, \ldots, \leqslant_d\}$ of linear extensions of $\leqslant$ on X such that

$$x \leqslant y \iff (x \leqslant_1 y) \wedge \cdots \wedge (x \leqslant_d y), \quad \text{for any } x, y \in X,$$

and the dimension is the minimum size of a realizer. The concept of dimension was introduced by Dushnik and Miller [5] and has been widely studied since. There are posets with arbitrarily large dimension: the *standard example* $S_k = (\{a_1, \ldots, a_k, b_1, \ldots, b_k\}, \leqslant)$, where $a_1, \ldots, a_k$ are minimal elements, $b_1, \ldots, b_k$ are maximal elements, and $a_i < b_j$ if and only if $i \neq j$, has dimension k when $k \geqslant 2$ [5]. On the other hand, the dimension of a poset is at most the width [9], and it is at most $\frac{n}{2}$ when $n \geqslant 4$, where n is the number of elements [9].

The *cover graph* of a poset $P = (X, \leqslant)$ is the graph on X with edge set $\{xy \colon x < y \text{ and there is no } z \text{ with } x < z < y\}$. A poset is *planar* if its cover graph has a non-crossing *upward drawing* in the plane, which means that every cover graph edge xy with $x < y$ is drawn as a curve that goes monotonically up from x to y. Planar posets that contain a least element and a greatest element are well known to have dimension at most 2 [1]. Trotter and Moore [17] proved that planar posets that contain a least element have dimension at most 3 (and so do posets whose cover graphs are forests) and asked whether all planar posets have bounded dimension. The answer is no—Kelly [13] constructed planar posets with arbitrarily large dimension (pictured). Another property of Kelly's posets is that their cover graphs have path-width and tree-width 3. Recent research brought a plethora of new bounds on dimension for structurally restricted posets. In particular, dimension is bounded for posets with



- height 2 and planar cover graphs [6],
- bounded height and planar cover graphs [16],
- bounded height and cover graphs of bounded tree-width [10],
- bounded height and cover graphs excluding a topological minor [18],
- bounded height and cover graphs of bounded expansion [12],
- cover graphs of path-width 2 [2] or, more generally, tree-width 2 [11].

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[3] Partially supported by National Science Center of Poland grant 2015/18/E/ST6/00299

Boolean dimension

The *boolean dimension* of a poset $P = (X, \leqslant)$ is the minimum number of linear orders on X a boolean combination of which gives $\leqslant$. More precisely, a *boolean realizer* of P is a set $\{\leqslant_1, \dots, \leqslant_d\}$ of linear orders on X for which there is a d-ary boolean formula ϕ such that

$$x \leqslant y \iff \phi\Big((x \leqslant_1 y), \dots, (x \leqslant_d y)\Big) \quad \text{for any } x, y \in X, \tag{1}$$

and the boolean dimension is the minimum size of a boolean realizer. The boolean dimension is at most the dimension, because a realizer is a boolean realizer for the formula $\phi(\alpha_1, \dots, \alpha_d) = \alpha_1 \wedge \dots \wedge \alpha_d$. Beware that the relation $\leqslant$ defined by (1) from arbitrary linear orders $\leqslant_1, \dots, \leqslant_d$ on X and formula ϕ is not necessarily a partial order.

Boolean dimension was first considered by Gambosi, Nešetřil, and Talamo [7] and by Nešetřil and Pudlák [15]. Boolean dimension d and dimension d are equivalent for $d \in \{1, 2, 3\}$ (this is essentially proved in [7], although the actual statement is more restricted), while the standard examples S_k with $k \geqslant 4$ have boolean dimension 4 [7]. The boolean dimension of an n-element poset can be as large as $\Theta(\log n)$ [15]. Nešetřil and Pudlák [15] asked whether boolean dimension is bounded for planar posets. It was proved already in [7] that posets with height 2 and planar cover graphs have bounded boolean dimension. This and the recent progress on dimension of structurally restricted posets have motivated revisiting boolean dimension in current research.

Local dimension

A *partial linear extension* of a partial order $\leqslant$ on X is a linear extension of the restriction of $\leqslant$ to some subset of X. A *local realizer* of P of *width* d is a set $\{\leqslant_1, \dots, \leqslant_t\}$ of partial linear extensions of $\leqslant$ such that every element of X occurs in at most d of $\leqslant_1, \dots, \leqslant_t$ and

$$x \leqslant y \iff \text{there is no } i \in \{1, \dots, t\} \text{ with } x >_i y, \quad \text{for any } x, y \in X. \tag{2}$$

The *local dimension* of P is the minimum width of a local realizer of P. Thus, instead of the size of a local realizer, we bound the number of times any element of X occurs in it. A set of linear extensions of $\leqslant$ is a local realizer if and only if it is a realizer. In particular, the local dimension is at most the dimension. For arbitrary partial linear extensions $\leqslant_1, \dots, \leqslant_t$ of $\leqslant$ on subsets of X, the relation $\leqslant$ defined by (2) is not necessarily a partial order—it may fail to be antisymmetric or transitive. It is antisymmetric, for example, if one of $\leqslant_1, \dots, \leqslant_t$ is a linear extension of $\leqslant$ on X. The concept of local dimension was proposed by Ueckerdt (Order & Geometry Workshop, Gułtowy, 2016) and originates from concepts studied in [3,14].

Results

Extending the results on boolean dimension from [7], for each d, we determine whether posets with dimension/boolean dimension/local dimension d have the other two parameters bounded or unbounded. Here is the full picture:

A *Boolean dimension d and dimension d are equivalent for $d \in \{1, 2, 3\}$ [7].*
B *Local dimension d and dimension d are equivalent for $d \in \{1, 2\}$.*
C *Standard examples have boolean dimension* 4 [7] *and local dimension* 3.
D *There are posets with boolean dimension* 4 *and unbounded local dimension.*
E *Posets with local dimension* 3 *have bounded boolean dimension.*
F *There are posets with local dimension* 4 *and unbounded boolean dimension.*

2 Proofs

B *Local dimension d and dimension d are equivalent for $d \in \{1, 2\}$.*

If a poset $P = (X, \leqslant)$ has local dimension 1, then its local realizer of width 1 must consist of a single full linear order on X. Now, let $P = (X, \leqslant)$ be a poset with local dimension 2, and consider a local realizer of P of width 2. If $x, y \in X$ are incomparable in $\leqslant$, then both occurrences of x and y are in the same two partial linear extensions, where $x < y$ in one and $x > y$ in the other. Therefore, the subposet of P induced on every connected component C of the incomparability graph of P is witnessed by two partial linear extensions, which restricted to C form a realizer of that subposet. These realizers stacked according to the order $\leqslant$ form a realizer of P of size 2.

C *Standard examples have boolean dimension* 4 *and local dimension* 3.

The standard example S_k has boolean dimension 4 (when $k \geqslant 4$), witnessed by the formula $\phi(\alpha) = \alpha_1 \wedge \alpha_2 \wedge (\alpha_3 \vee \alpha_4)$ and the following four linear orders:

$$a_1 < \dots < a_k < b_1 < \dots < b_k, \qquad b_1 < a_1 < \dots < b_k < a_k,$$
$$a_k < \dots < a_1 < b_k < \dots < b_1, \qquad b_k < a_k < \dots < b_1 < a_1.$$

It has local dimension 3 (when $k \geqslant 3$), witnessed by the two linear extensions above on the left and k partial linear extensions each of the form $b_i < a_i$.

D *There are posets with boolean dimension* 4 *and unbounded local dimension.*

Another well-known construction of posets with arbitrarily large dimension [5] involves incidence posets of complete graphs: $P_n = (V \cup E, \leqslant)$, where $V = \{v_1, \ldots, v_n\}$ are the minimal elements, $E = \{v_1v_2, v_1v_3, \ldots, v_{n-1}v_n\}$ are the maximal elements, and the only comparable pairs are $v_i < v_iv_j$ and $v_j < v_iv_j$ for $i \neq j$. The boolean dimension of P_n is at most 4, witnessed by the formula $\phi(\alpha) = (\alpha_1 \wedge \alpha_2) \vee (\alpha_3 \wedge \alpha_4)$ and the following four linear orders:

$$\begin{aligned}
&A_1 < \cdots < A_n, \text{ where each } A_i \text{ has form } v_i < v_iv_{i+1} < \cdots < v_iv_n,\\
&B_n < \cdots < B_1, \text{ where each } B_i \text{ has form } v_i < v_iv_n < \cdots < v_iv_{i+1},\\
&C_1 < \cdots < C_n, \text{ where each } C_i \text{ has form } v_i < v_1v_i < \cdots < v_{i-1}v_i,\\
&D_n < \cdots < D_1, \text{ where each } D_i \text{ has form } v_i < v_{i-1}v_i < \cdots < v_1v_i.
\end{aligned}$$

The local dimension of P_n is unbounded as $n \to \infty$. For suppose P_n has a local realizer of width d. Enumerate the occurrences of each element of $V \cup E$ in the local realizer from 1 to (at most) d. Each triple $v_iv_jv_k$ $(i < j < k)$ can be assigned a color (p, q) so that $v_iv_k < v_j$ in a partial linear extension containing the pth occurrence of v_j and the qth occurrence of v_iv_k. By Ramsey's theorem, if n is large enough, then there is a quadruple $v_iv_jv_kv_\ell$ $(i < j < k < \ell)$ with all four triples of the same color (p, q). Therefore, the pth occurrences of v_j and v_k and the qth occurrences of v_iv_ℓ, v_iv_k, and v_jv_ℓ are all in the same partial linear extension, which contains a cycle $v_j < v_jv_\ell < v_k < v_iv_k < v_j$, a contradiction.

E *Posets with local dimension* 3 *have bounded boolean dimension.*

Let $P = (X, \leqslant)$ be a poset with a local realizer of width 3 consisting of partial linear extensions that we call *gadgets*. We construct a boolean realizer $\{\leqslant^\star, \leqslant_1, \leqslant'_1, \ldots, \leqslant_d, \leqslant'_d\}$ for a formula of the form $\alpha^\star \wedge (\alpha_1 \vee \alpha'_1) \wedge \cdots \wedge (\alpha_d \vee \alpha'_d)$. The order $\leqslant^\star$ is an arbitrary linear extension of $\leqslant$ on X. Each pair $\leqslant_i, \leqslant'_i$ is defined by $X_1 <_i \cdots <_i X_t$ and $X_t <'_i \cdots <'_i X_1$, where $\{X_1, \ldots, X_t\}$ is some partition of X into *blocks* such that every block X_j is completely ordered by some gadget and that order is inherited by $\leqslant_i$ and $\leqslant'_i$. It remains to construct a bounded number of partitions of X into blocks so that for any $x, y \in X$, if $x <^\star y$ and $x > y$ in some gadget, then $x > y$ in some block.

Enumerate the occurrences of each $x \in X$ in the gadgets as x^1, x^2, x^3 according to a fixed order of the gadgets. For each $p \in \{1, 2, 3\}$, form a partition of X by restricting every gadget to the x^ps. These three partitions witness all comparabilities of the form $x^p > y^p$ within gadgets. Now, let G be a graph on X where xy (with $x <^\star y$) is an edge if and only if $x^p > y^q$ $(p \neq q)$ in some gadget. The fact that every element of X has only 3 occurrences in the gadgets implies that G has bounded chromatic number (we omit the details). Let c be a coloring of G with a bounded number of colors. For $1 \leqslant p < q \leqslant 3$ and any distinct colors a, b, form a partition of X by restricting every gadget to the x^ps with $c(x) = a$ and the y^qs with $c(y) = b$ (adding singletons if necessary to obtain a full partition of X). These partitions witness all comparabilities of the form $x^p > y^q$ (where $x <^\star y$ and $p \neq q$) within gadgets. The overall number of partitions thus obtained is bounded.

F *There are posets with local dimension* 4 *and unbounded boolean dimension.*

When (V, E) is an acyclic digraph and $v \in V$, let $E^+(v) = \{uv \in E\colon u \in V\}$ and $E^-(v) = \{vw \in E\colon w \in V\}$ (uv denotes a directed edge from u to v). For $k \geqslant 1$, we construct an acyclic digraph $G = (V, E)$ with $\chi(G) > k$, a poset $P = (E, \leqslant)$, and its local realizer $\{\leqslant_A, \leqslant_B\} \cup \{\leqslant_v\colon v \in V\}$ of width 4, where

- $\leqslant_A$ and $\leqslant_B$ are (full) linear extensions of $\leqslant$ on E such that $E^+(v) <_A E^-(v)$ and $E^+(v) <_B E^-(v)$ for every $v \in V$,
- $\leqslant_v$ is a partial linear extension of the form $E^-(v) <_v E^+(v)$, for each $v \in V$.

This is achieved by using Tutte's construction of a triangle-free graph G with $\chi(G) > k$ [4]. We omit the details, noting that the main challenge is to ensure transitivity of the relation $\leqslant$ defined by (2) (it could not be achieved when G was, for instance, a transitive tournament). We show that when $k = 2^{2^{2^d}}$, the resulting poset P has boolean dimension greater than d. For suppose $\{\leqslant_1, \ldots, \leqslant_d\}$ is a boolean realizer of P for a formula ϕ. Let $H = (A, B)$, where $A = \{uvw\colon uv, vw \in E\}$ and $B = \{uvwx\colon uvw, vwx \in A\}$. It follows that $\chi(H) \geqslant \log_2 \log_2 \chi(G) > 2^d$ [8, Theorem 9]. For $uvw \in A$, let $\alpha(uvw) = ((uv <_1 vw), \ldots, (uv <_d vw)) \in \{0, 1\}^d$; since $uv >_v vw$, we have $uv \not\leqslant vw$ and thus $\phi(\alpha(uvw)) = 0$. Let $uvwx \in B$. We have $uv <_A vw <_A wx$ and $uv <_B vw <_B wx$, which implies $uv < wx$, because no partial linear extension from $\{\leqslant_v\colon v \in V\}$ contains both uv and wx. If $\alpha(uvw) = \alpha(vwx) = \alpha$, then transitivity of $\leqslant_1, \ldots, \leqslant_d$ implies $((uv <_1 wx), \ldots, (uv <_d wx)) = \alpha$. This, $\phi(\alpha) = 0$, and $uv < wx$ result in a contradiction. Therefore, $\alpha\colon A \to \{0, 1\}^d$ is a 2^d-coloring of H. This contradicts the fact that $\chi(H) > 2^d$.

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