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Separating tree-chromatic number from path-chromatic number

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A R T I C L E I N F O

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We apply Ramsey theoretic tools to show that there is a family of graphs which have tree-chromatic number at most 2 while the path-chromatic number is unbounded. This resolves a problem posed by Seymour.

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1. Introduction

Let G be a graph. A *tree-decomposition* of G is a pair $(T, \mathcal{B})$ where T is a tree and $\mathcal{B} = (B_t \mid t \in V(T))$ is a family of subsets of $V(G)$, satisfying:

(T1) for each $v \in V(G)$ there exists $t \in V(T)$ with $v \in B_t$; and for every edge $uv \in E(G)$ there exists $t \in V(T)$ with $u, v \in B_t$;

(T2) for each $v \in V(G)$, if $v \in B_t \cap B_{t''}$ for some $t, t'' \in V(T)$, and t' lies on the path in T between t and t'', then $v \in B_{t'}$.

Many researchers refer to the subset B_t as a *bag* and they consider B_t as an induced subgraph of G. With this convention, $|B_t|$ is just the number of vertices of G in the bag B_t, while $\chi(B_t)$ is the chromatic number of the induced subgraph of G determined by the vertices in B_t.

The quality of a tree-decomposition $(T, (B_t \mid t \in V(T)))$ is usually measured by its *width*, i.e. the maximum of $|B_t| - 1$ over all $t \in V(T)$. Then the *tree-width* of G is the minimum width of a tree-decomposition of G. In this paper we study the tree-chromatic number of a graph, a concept introduced by Seymour in [5]. The *chromatic number* of a tree-decomposition $(T, (B_t \mid t \in V(T)))$ is the maximum of $\chi(B_t)$ over all $t \in V(T)$. The *tree-chromatic number* of G, denoted by tree-$\chi(G)$, is the minimum chromatic number of a tree-decomposition of G. A tree-decomposition $(T, (B_t \mid t \in V(T)))$ is a *path-decomposition* when T is a path. The *path-chromatic number* of G, denoted by path-$\chi(G)$, is the minimum chromatic number of a path-decomposition of G. Clearly, for every graph G we have

$$\omega(G) \leqslant \text{tree-}\chi(G) \leqslant \text{path-}\chi(G) \leqslant \chi(G).$$

Furthermore, if $G = K_n$ is the complete graph on n vertices, then $\omega(G) = \chi(G) = n$, so all these inequalities can be tight. Accordingly, it is of interest to ask whether for consecutive parameters in this series of inequalities, there is a sequence of graphs for which one parameter is bounded while the next parameter is unbounded.

In [5], Seymour proved that the classic Erdős construction [1] for graphs with large girth and large chromatic number yields a sequence $\{G_n : n \geqslant 1\}$ with $\omega(G_n) = 2$ and tree-$\chi(G_n)$ unbounded.

For an integer $n \geqslant 2$, the *shift graph* S_n is a graph whose vertex set consists of all closed intervals of the form $[a, b]$ where a, b are integers with $1 \leqslant a < b \leqslant n$. Vertices $[a, b]$, $[c, d]$ are adjacent in S_n when $b = c$ or $d = a$. As is well known (and first shown in [2]), $\chi(S_n) = \lceil \lg n \rceil$, for every $n \geqslant 2$. On the other hand, S_n has a natural simple path decomposition. T is simply the path on the vertices $t_1, t_2 \ldots, t_n$ and for $1 \leqslant i \leqslant n$ we have $B_{t_i} = \{[a, b] \in V(S_n) \ : \ a \leqslant i \leqslant b\}$. Then for every $1 \leqslant i \leqslant n$ the bag B_{t_i} is the union of two independent sets, namely $\{[a, b] \in V(S_n) \ : \ a < i \leqslant b\}$ and $\{[a, b] \in V(S_n) \ : \ a \leqslant i < b\}$, and hence the chromatic number of the corresponding induced subgraph is at most 2. This shows that path-$\chi(S_n) \leqslant 2$ for every $n \geqslant 2$, so as noted in [5], the family of shift graphs has bounded path-chromatic number and unbounded chromatic number.

Accordingly, it remains only to determine whether there is an infinite sequence of graphs with bounded tree-chromatic number and unbounded path-chromatic number. However, these two parameters appear to be more subtle in nature. As a first step, Huynh and Kim [3] showed that there is an infinite sequence $\{G_n : n \geqslant 1\}$ of graphs with tree-$\chi(G_n) \to \infty$ and tree-$\chi(G_n) <$ path-$\chi(G_n)$ for all $n \geqslant 1$.

In [5], Seymour proposed the following construction. Let T_n be the complete (rooted) binary tree with 2^n leaves. When y and z are distinct vertices in T_n, the path from y to z is called a "V" when the unique point on the path which is closest to the root of T_n is an intermediate point x on the path which is *strictly* between y and z. We refer to x as the *low point* of the V formed by y and z.

For a fixed value of n, we can then form a graph G_n whose vertices are the V's in T_n. We take V adjacent to V' in G_n when an end point of one of the two paths is the low point of the other. Clearly, $\omega(G_n) \leqslant 2$. Furthermore, it is easy to see that $\chi(G_n) \to \infty$ with n (we will say more about this observation later in the paper), and Seymour [5] suggested that the family $\{G_n : n \geqslant 1\}$ has unbounded path-chromatic number.

However, we will show that graphs in the family $\{G_n : n \geqslant 1\}$ have bounded path-chromatic number. In fact, we will use Ramsey theoretic tools developed by Milliken [4] to show that if we fix $r \geqslant 2$, and assume we have a path-decomposition of G_n witnessing that path-$\chi(G_n) \leqslant r$, then this decomposition is (essentially) uniquely determined. Furthermore, this decomposition actually witnesses that path-$\chi(G_n) \leqslant 2$.

Moreover, in analyzing this decomposition, we discovered the following minor modification. In the binary tree T_n, a subtree is called a "Y" when it has 3 leaves and the closest vertex in the subtree to the root of T_n is one of the three leaves. We then let H_n be the graph whose vertex set consists of the V's and Y's in T_n. Furthermore, Y is adjacent to Y' in H_n if and only if one of the two upper leaves of one of them is the lowest leaf in the other. Also, a Y is adjacent to a V if and only if one of the upper leaves in the Y is the low point of the V.

It is clear from the natural tree-decomposition of H_n that tree-$\chi(H_n) \leqslant 2$. Using Ramsey theoretic tools, we will then be able to show that path-$\chi(H_n) \to \infty$ with n, so that Seymour's question has been successfully resolved.

2. Ramsey theory on binary trees

The Ramsey theoretic concepts discussed here are treated in a more comprehensive manner by Milliken [4],[5] but we will find it convenient to use somewhat different notation and terminology.

[5] The particular result we need is Theorem 2.1 on page 220. Note that Milliken credits the result to Halpern, Läuchi, Laver and Pincus and comments on the history of the result.

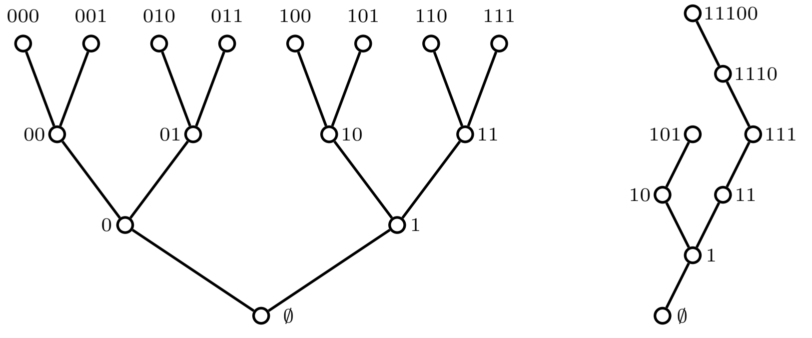

Fig. 1. Binary trees: down sets in T_n.

For a positive integer n, we view the *complete binary tree* T_n as the poset[6] consisting of all binary strings of length at most n, with $x \leqslant y$ in T_n when x is an initial segment in y. The empty string, denoted $\emptyset$, is then the zero (least element) of T_n. For all $n \geqslant 1$, T_n has $2^{n+1} - 1$ elements and height $n+1$. In particular, T_0 is the one-point poset whose only element is the empty string.

When $n \geqslant 1$ and x is a binary string of length n, we will denote coordinate i of x as $x(i)$ and when a string is of modest length, we may write it explicitly, e.g., $x = 01001101$. When $n \geqslant m > p \geqslant 0$, x is a string of length p, y is a string of length m and $x < y$ in T_n, we say y is in the *left tree above* x when $y(p+1) = 0$ and we say y is in the *right tree above* x when $y(p+1) = 1$.

Recall that in a poset P, a subposet Q of P is called a *down set* if $x \in Q$ whenever $y \in Q$ and $x \leqslant y$ in P. We will refer to down sets of the complete binary tree T_n as *binary trees*. In Fig. 1, we show on the left the complete binary tree T_3. On the right, we show a binary tree Q which will be a down set in any complete binary tree T_n with $n \geqslant 5$.

Let $n \geqslant 0$, let Q be a binary tree in T_n, and let R be a subposet of T_n. Following Milliken [4], we will say R is a *strong copy of* Q when there is a function $f : Q \to R$ satisfying the following two requirements:

(i) f is a poset isomorphism, i.e., f is a bijection and for all $x, y \in Q$, $x \leqslant y$ in Q if and only if $f(x) \leqslant f(y)$ in R.
(ii) For all $x, y \in Q$ with $x < y$ in Q, y is in the left tree above x in Q if and only if $f(y)$ is in the left tree above $f(x)$ in T_n.

Since we are concerned with binary trees, we note that when f satisfies the preceding two conditions, then it automatically implies that y is in the right tree above x if and only if $f(y)$ is in the right tree above $f(x)$.

[6] The complete (rooted) binary tree we discussed in an informal manner in the opening section of this paper is just the cover graph of the poset T_n defined here.

For the remainder of this paper, when $r \geqslant 1$, we let $[r]$ denote the set $\{1, 2, \ldots, r\}$. Also, an r-coloring of a set X is just a map $\Phi : X \to [r]$. In some situations, we will consider a coloring Φ using a set of r colors, but the set will not simply be the set $[r]$.

The following result is a straightforward extension of the special case of Theorem 2.1 from [4] for binary trees.

Theorem 2.1. *For every triple (Q, p, r), where Q is a binary tree, and p and r are positive integers with p at least as large as the height of Q, there is a least positive integer $n_0 = \mathrm{Ram}(Q, p, r)$ so that if $n \geqslant n_0$ and Φ is an r-coloring of the strong copies of Q in T_n, then there is a color $\alpha \in [r]$ and a subposet R of T_n such that R is a strong copy of T_p and Φ assigns color α to every strong copy of Q contained in R.*

3. Separating tree-chromatic number and path-chromatic number

For the remainder of the paper, for a positive integer n, we let G_n be the graph of the V's in the complete binary tree T_n. Strictly speaking, a vertex V in G_n is a path which is determined by its two endpoints, but we find it convenient to specify V as a triple (x, y, z), where y and z are the endpoints of the path and x is the low point on the path. We view V as a triple and not a 3-element set so we can follow the convention that y is in the left tree above x and z is in the right tree above x. When $V_1 = (x_1, y_1, z_1)$ and $V_2 = (x_2, y_2, z_2)$ are vertices in G_n, we note that V_1 and V_2 are adjacent if and only if one of the following four statements holds: $z_1 = x_2$, $y_1 = x_2$, $y_2 = x_1$ or $z_2 = x_1$.

Also, for each $n \geqslant 1$, we let H_n be the graph of V's and Y's in T_n. Of course, G_n is an induced subgraph of H_n. Furthermore, the natural tree-decomposition of H_n shows that tree-$\chi(H_n) \leqslant 2$ for all $n \geqslant 1$.

Our goals for this section are to prove the following two theorems.

Theorem 3.1. *For all $n \geqslant 1$, the path-chromatic number of the graph G_n of V's in the complete binary tree T_n is at most 2.*

Theorem 3.2. *For every positive integer r, there is a least positive integer n_0 so that if $n \geqslant n_0$, then the path-chromatic number of the graph H_n of V's and Y's in the complete binary tree T_n has chromatic number larger than r.*

We elect to follow the line of our research and prove the second of these two theorems first. In accomplishing this goal, we will discover a path-decomposition of G_n witnessing that path-$\chi(G_n) \leqslant 2$ for all $n \geqslant 1$.

Our argument for Theorem 3.2 will proceed by contradiction, i.e. we will assume that there is some positive integer r such that path-$\chi(H_n) \leqslant r$ for all $n \geqslant 1$. The contradiction will come when n is sufficiently large in comparison to r.

For the moment, we take n as a large but unspecified integer. Later, it will be clear how large n needs to be. We then take a path-decomposition of H_n witnessing that path-$\chi(H_n) \leqslant r$. We may assume that the host path in this decomposition is the set $\mathbb{N}$

of positive integers with i adjacent to $i+1$ in $\mathbb{N}$ for all $i \geqslant 1$. For each vertex v in H_n, the set of all integers i for which $v \in B_i$ is a set of consecutive integers, and we denote the least integer in this set as a_v and the greatest integer as b_v. Abusing notation slightly, we will denote this set as $[a_v, b_v]$, i.e., this interval notation identifies the integers $i \in \mathbb{N}$ with $a_v \leqslant i \leqslant b_v$. Alternatively, $[a_v, b_v]$ is just the set of integers i for which v is in the bag B_i. We point out the requirement that $[a_v, b_v] \cap [a_u, b_u] \neq \emptyset$ when v and u are adjacent vertices in H_n.

After duplicating vertices in the path decomposition if necessary, we may assume that $a_v < b_v$ for every vertex $v \in V(H_n)$. Similarly, after adding extra vertices to the path decomposition if necessary, we may assume that for each integer i, there is at most one vertex $v \in V(H_n)$ with $i \in \{a_v, b_v\}$.

For each $i \in \mathbb{N}$, we let $G_n(i)$ denote the induced subgraph of G_n determined by those vertices $v \in G_n$ with $i \in [a_v, b_v]$. Alternatively, $G_n(i)$ is the subgraph of G_n induced by the vertices in bag B_i. The graph $H_n(i)$ is defined analogously.

We pause here to point out an essential detail for the remainder of the proof. Since $\chi(G_n(i)) \leqslant \chi(H_n(i)) \leqslant r$ for all integers i, then for all $q > 2^r$, there is no positive integer i for which either $G_n(i)$ or $H_n(i)$ contains the shift graph S_q as a subgraph.

To begin to make the connection with Ramsey theory, we observe that there is a natural 1–1 correspondence between V's in G_n and strong copies of T_1 in T_n. So in the discussion to follow, we will interchangeably view a vertex $V = (x, y, z)$ of G_n as a path in T_n and as a 3-element subposet of T_n forming a strong copy of T_1. Of course, we are abusing notation slightly by referring to T_n as a graph and as a poset, but by now the notion that as a graph, we are referring to the cover graph of the poset should be clear.

In the discussion to follow, when we discuss a family $\{V_j : j \in [t]\}$ of V's in G_n, we will let $V_j = (x_j, y_j, z_j)$, and we will let $[a_j, b_j]$ be the interval in the path-decomposition corresponding to V_j, for each $j \in [t]$.

Let (V_1, V_2) be an ordered pair of vertices in G_n. Referring to the binary trees in Fig. 2, we consider the 7 different ways this pair can appear in T_n so that the two paths V_1 and V_2 have at most one vertex from T_n in common:

(i) V_1 and V_2 are adjacent with $z_1 = x_2$. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_1.

(ii) V_1 and V_2 are adjacent with $y_1 = x_2$. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_2.

(iii) V_1 and V_2 are non-adjacent with x_2 in the right tree above z_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_3.

(iv) V_1 and V_2 are non-adjacent with x_2 in the left tree above y_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_4.

(v) V_1 and V_2 are non-adjacent with x_2 in the left tree above z_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_5.

(vi) V_1 and V_2 are non-adjacent with x_2 in the right tree above y_1. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_6.

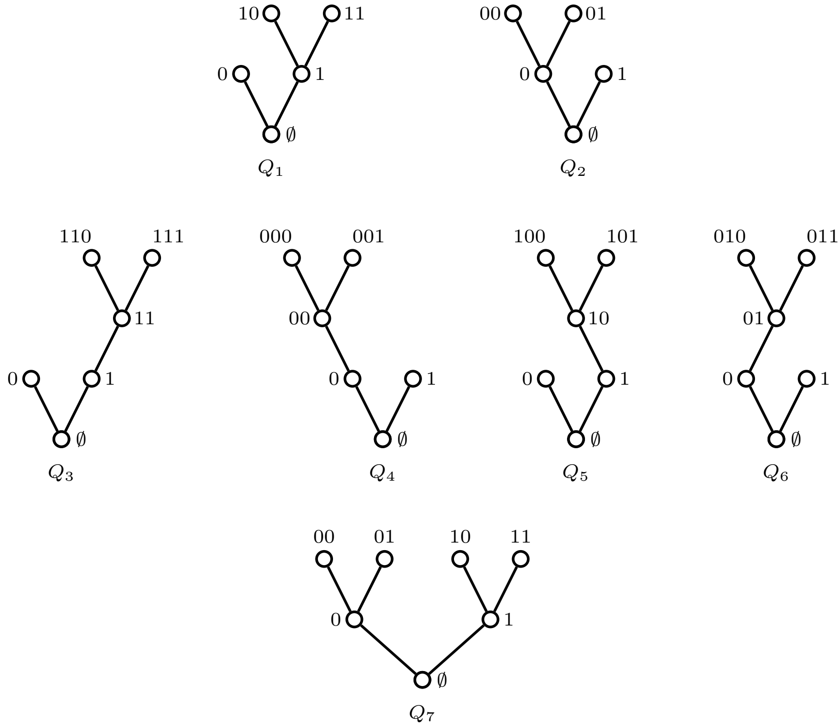

Fig. 2. Applying Ramsey with seven binary trees.

(vii) V_1 and V_2 are non-adjacent and there is a vertex w in T_n so that x_1 is in the left tree above w while x_2 is in the right tree above w. In this case, we associate the pair (V_1, V_2) with a strong copy of the poset Q_7.

Also, given a pair (V_1, V_2) of distinct V's in G_n, there are 6 ways the intervals $[a_1, b_1]$ and $[a_2, b_2]$ can appear in the path-decomposition:

$a_1 < a_2 < b_1 < b_2$	Overlapping, moving right
$a_2 < a_1 < b_2 < b_1$	Overlapping, moving left
$a_1 < b_1 < a_2 < b_2$	Disjoint, moving right
$a_2 < b_2 < a_1 < b_1$	Disjoint, moving left
$a_1 < a_2 < b_2 < b_1$	Inclusion, second in first
$a_2 < a_1 < b_1 < b_2$	Inclusion, first in second

In the arguments to follow, we will abbreviate these 6 options as OMR, OML, DMR, DML, ISF and IFS, respectively.

We then define for each $i \in [7]$ a 6-coloring Φ_i of the strong copies of Q_i in T_n. The colors will be the six labels $\{\text{OMR}, \text{OML}, \ldots, \text{IFS}\}$ listed above. When $i \in [7]$ and Q is a

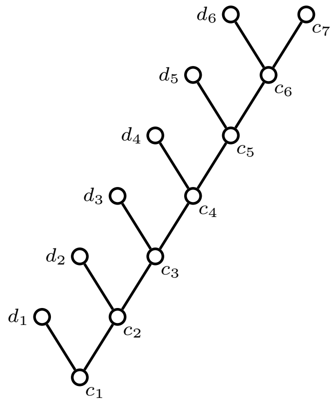

Fig. 3. A shift graph in G_n.

strong copy of Q_i, then Q is associated with a pair (V_1, V_2) of vertices from G_n. It is then natural to set $\Phi_i(Q)$ as the label describing how the pair $([a_1, b_1], [a_2, b_2])$ of intervals are positioned in the path decomposition.

Now let $p = 4 \cdot 2^r$. By iterating on Theorem 2.1, we may assume that n is sufficiently large to guarantee that there is a subposet R of T_n and a vector $(\alpha_1, \alpha_2, \ldots, \alpha_7)$ of colors such that R is a strong copy of T_p and for each $i \in [7]$, Φ_i assigns color α_i to all strong copies of Q_i in R. In the remainder of the argument, we will abuse notation slightly and simply consider that $R = T_p$.

Claim 1. α_1 *is either* OMR *or* OML*.*

Proof. A pair (V_1, V_2) of vertices in G_n associated with a strong copy of Q_1 in T_p is adjacent in G_n so that $[a_1, b_1]$ and $[a_2, b_2]$ intersect. So α_1 cannot be DMR or DML. We assume that α_1 is ISF and argue to a contradiction. The argument when α_1 is IFS is symmetric. Consider the subposet of T_p consisting of all non-empty strings for which each bit, except possibly the last, is a 1. We suggest how this subposet appears (at least for a modest value) in Fig. 3.

Using the labelling given in Fig. 3, for each interval $[i, j]$ with $1 \leqslant i < j \leqslant p$, we consider the vertex $V[i, j] = (c_i, d_i, c_j)$. Clearly, $V[i, j]$ is adjacent to $V[j, k]$ when $1 \leqslant i < j < k \leqslant p$, i.e., these vertices form the shift graph S_p.

Let $[a, b] = [a_{V[p-1,p]}, b_{V[p-1,p]}]$ be the interval for the vertex $V[p-1, p]$. We claim that for each $[i, j]$ with $1 \leqslant i < j \leqslant p-1$, the interval for $V[i, j]$ in the path-decomposition for H_n contains $[a, b]$. This is immediate if $j = p-1$, since $(V[i, p-1], V[p-1, p])$ is assigned color ISF. Now suppose $j < p-1$. Then $(V[i, j], V[j, p-1])$ is also ISF, so that in the path-decomposition, the interval for $V[j, p-1]$ is included in the interval for $V[i, j]$. By transitivity, we conclude that $[a, b]$ is included in the interval for $V[i, j]$.

So the V's in $\{V[i, j] : 1 \leqslant i < j \leqslant p-1\}$ form a copy of the shift graph S_{p-1}, and all of them are in the bag $G_n(a)$. Since $p = 4 \cdot 2^r$, this is a contradiction. □

Without loss of generality, we take α_1 to be OMR, since if α_1 is OML, we may simply reverse the entire path-decomposition. To help keep track of the configuration information as it is discovered, we list this statement as a property.

Property 1. $\alpha_1 =$ OMR*, i.e.,* Φ_1 *assigns color* OMR *to a pair* (V_1, V_2) *of adjacent vertices in* G_n *when* $z_1 = x_2$*.*

Although it may not be a surprise, once the color α_1 is set, colors $\alpha_2, \alpha_3, \dots, \alpha_7$ are determined.

Property 2. $\alpha_3 =$ DMR*, i.e.,* Φ_3 *assigns color* DMR *to a pair* (V_1, V_2) *of non-adjacent vertices in* G_n *when* x_2 *is in the right tree above* z_1*.*

Proof. Let (V_1, V_2) be a pair of non-adjacent vertices in G_n with x_2 in the right tree above z_1. Then let w_3 be the string formed by attaching a 0 at the end of z_1, and set $V_3 = (z_1, w_3, x_2)$. Then V_3 is adjacent to both V_1 and V_2. Furthermore, $\Phi_1(V_1, V_3) =$ OMR and $\Phi_1(V_3, V_2) =$ OMR. Accordingly, α_3 is either OMR or DMR. We assume that $\alpha_3 =$ OMR and argue to a contradiction.

Consider the shift graph used in the proof of Claim 1. Let $a = a_{V[p-1,p]}$ be the left endpoint of the interval for $V[p-1,p]$ in the path-decomposition. We claim that a is in the interval for $V[i,j]$ in the path-decomposition whenever $1 \leqslant i < j \leqslant p-1$. Again, this holds when $j = p-1$ since $\Phi_1(V[i,p-1], V[p-1,p]) =$ OMR. Also, when $j < p-1$, the color assigned by Φ_3 to the pair $(V[i,j], V[p-1,p])$ is also OMR, so that the interval for $V[i,j]$ in the path-decomposition also contains a. This now implies that $G_n(a)$ contains the shift graph S_{p-1}. The contradiction completes the proof. □

Property 3. $\alpha_2 =$ OML*, i.e.,* Φ_2 *assigns color* OML *to a pair* (V_1, V_2) *of adjacent vertices in* G_n *when* $y_1 = x_2$*. Also,* $\alpha_4 =$ DML*, i.e.,* Φ_4 *assigns color* DML *to a pair* (V_1, V_2) *of non-adjacent vertices in* G_n *when* x_2 *is in the left tree above* y_1*.*

Proof. We can repeat the arguments given previously to conclude that one of two cases must hold: Either (1) $\alpha_2 =$ OMR and $\alpha_4 =$ DMR, or (2) $\alpha_2 =$ OML and $\alpha_4 =$ DML. We assume that $\alpha_2 =$ OMR and $\alpha_4 =$ DMR and argue to a contradiction. Consider the binary tree contained in T_p as shown on the left side of Fig. 4. Let $V_1 = (f, g, h)$, $V_2 = (i, j, k)$, $V_3 = (c, f, e)$ and $V_4 = (c, d, i)$.

Since $\Phi_4(V_4, V_1) =$ DMR, we know $b_4 < a_1$. Since $\Phi_1(V_4, V_2) =$ OMR, we know $a_2 < b_4$, so $a_2 < a_1$. Since $\Phi_3(V_3, V_2) =$ DMR, we know $b_3 < a_2$ so $b_3 < a_1$. But $\Phi_2(V_3, V_1) =$ OMR, which requires $a_1 < b_3$. The contradiction completes the proof of Property 3. □

Property 4. $\alpha_7 =$ DMR*, i.e.,* Φ_7 *assigns color* DMR *to a pair* (V_1, V_2) *of non-adjacent vertices in* G_n *when there is a vertex* w *in* T_n *such that* x_1 *is in the left tree above* w *while* x_2 *is in the right tree above* w*.*

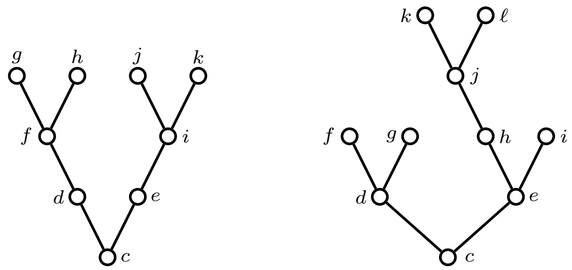

Fig. 4. Two useful small examples.

Proof. We again consider the binary tree shown on the left side of Fig. 4. Again, we take $V_1 = (f, g, h)$ and $V_2 = (i, j, k)$. Noting that f is in the left tree above c and i is in the right tree above a, $\Phi_7(V_1, V_2) = \alpha_7$.

Now let $V_5 = (c, d, e)$. Then $\Phi_4(V_5, V_1) = \text{DML}$ and $\Phi_3(V_5, V_2) = \text{DMR}$. These statements imply $\alpha_7 = \text{DMR}$. □

Property 5. $\alpha_5 = \alpha_6 = \text{ISF}$, *i.e., Φ_5 assigns color* ISF *to a pair (V_1, V_2) of non-adjacent vertices in G_n when x_2 is in the left tree above z_1 and Φ_6 assigns this pair color* IFS *when x_2 is in the right tree above y_1.*

Proof. We prove that $\alpha_5 = \text{ISF}$. The argument to show that $\alpha_6 = \text{ISF}$ is symmetric. Consider the binary tree shown on the right side of Fig. 4. Let $V_1 = (c, d, e)$ and $V_2 = (j, k, l)$. Then j is in the left tree above e, so $\Phi_5(V_1, V_2) = \alpha_5$.

Now set $V_3 = (d, f, g)$ and $V_4 = (e, h, i)$. We observe that $\Phi_2(V_1, V_3) = \text{OML}$, $\Phi_7(V_3, V_2) = \text{DMR}$, $\Phi_1(V_1, V_4) = \text{OMR}$ and $\Phi_4(V_4, V_2) = \text{DML}$. Together, these statements imply $\alpha_5 = \text{ISF}$. □

Up to this point in the proof, our entire focus has been on the V's in G_n. We now turn our attention to properties that the Y's in H_n must satisfy.

Consider the binary tree shown in Fig. 5. Of course, we intend that this tree appear inside T_p. In our figure, the "size" of this construction is $m = 6$, but since $p = 4 \cdot 2^r$, we know we can make $m > 2^r$. For each interval $[i, j]$ with $1 \leqslant i < j \leqslant m$, we let $Y[i, j]$ be the Y whose three leaves are x_i, x_j and w_j. Clearly, the family $\{Y[i, j] : 1 \leqslant i < j \leqslant m\}$ forms a copy of the shift graph S_m. To reach a final contradiction, it remains only to show that there is some integer $k \in \mathbb{N}$ for which all vertices in $\{Y[i, j] : 1 \leqslant i < j \leqslant m\}$ belong to $H_n(k)$.

For each $j \in [m]$, we let $V_j = (x_j, y_j, z_j)$, and as usual, we let $[a_j, b_j]$ be the corresponding interval for V_j in the path decomposition. By Property 2, we have $\alpha_3 = \text{DMR}$, so that:

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{m-1} < b_{m-1} < a_m < b_m.$$

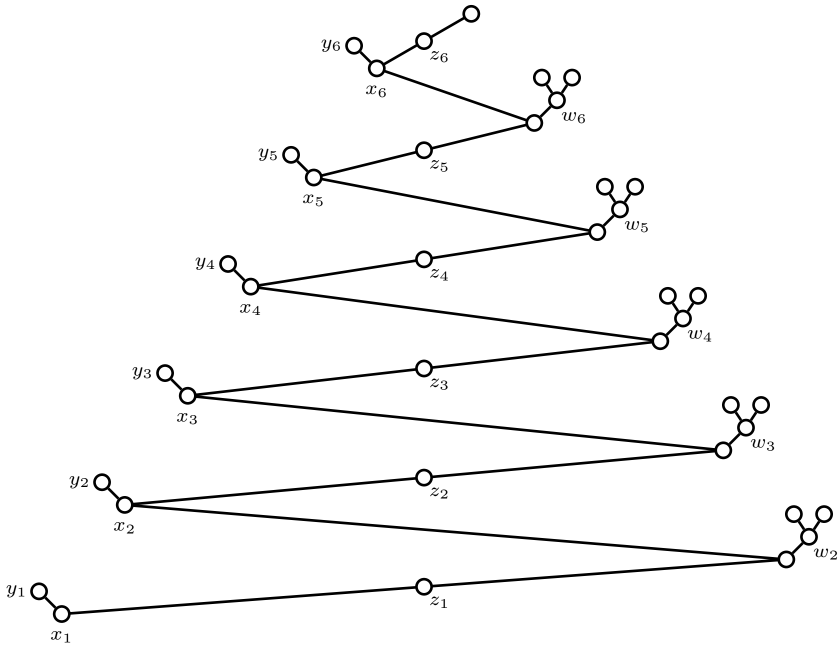

Fig. 5. The final counter-example.

For each $j = 2, 3, \ldots, m$, let $V'_j = (w_j, w_j0, w_j1)$, and we let $[a'_j, b'_j]$ be the corresponding interval in the path-decomposition. By Property 4, $\alpha_7 = \text{DMR}$ so that:

$$a'_m < b'_m < a'_{m-1} < b'_{m-1} < \cdots < a'_3 < b'_3 < a'_2 < b'_2.$$

Again, since $\alpha_7 = \text{DMR}$, we know that $a_m < b_m < a'_m < b'_m$.

Now consider a pair i, j with $1 \leqslant i < j \leqslant m$. The vertex $Y[i, j]$ is adjacent in H_n to both V_j and V'_j. This implies that the interval for $Y[i, j]$ must overlap both $[a_j, b_j]$ and $[a'_j, b'_j]$. However, this forces the interval for $Y[i, j]$ to contain $[b_m, a'_m]$. Therefore, $H_n(b_m)$ contains the shift graph S_m. With this observation, the proof of Theorem 3.2 is complete.

We now return to the task of proving Theorem 3.1, i.e., the assertion that path-$\chi(G_n) \leqslant 2$ for all $n \geqslant 1$. Our proof for Theorem 3.2 suggests a natural way to define a path-decomposition of the graph G_n of V's in the binary tree T_n, one that satisfies all five properties we have developed to this point. We simply take a drawing in the plane of T_n using a geometric series approach. Taking a standard cartesian coordinate system in the plane, we place the zero of T_n at the origin. If $m \geqslant 0$ and we have placed a string x of length m at (h, v), we set $\delta = 2^{-m}$ and place $x1$ and $x0$ at $(h + \delta, v + \delta)$ and $(h - \delta, v + \delta)$, respectively.

For each x in T_n, let $\pi(x)$ denote the vertical projection of x down onto the horizontal axis. In turn, for each $V = (x, y, z)$, we take $a_V = \pi(y)$ and $b_V = \pi(z)$. To illustrate

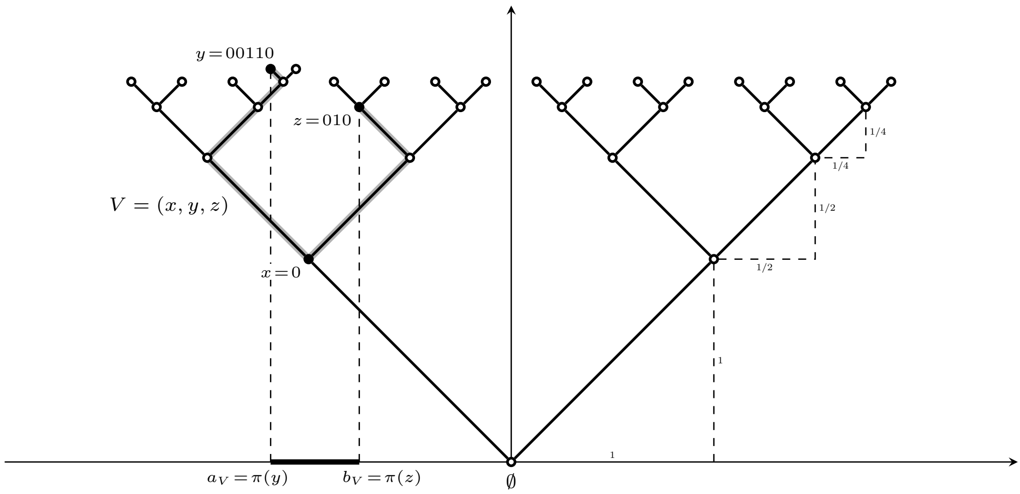

Fig. 6. A path-decomposition of G_n.

this construction, we show in Fig. 6 the interval $[a_V, b_V]$ corresponding to the vertex $V = (0, 00110, 010)$ in G_n.

Clearly, we may consider the host path P for the decomposition as consisting of all points on the horizontal axis of the form $\pi(x)$ where $x \in T_n$. Also, in the natural manner, $\pi(x)$ is adjacent to $\pi(x')$ in P when there is no string $x'' \in T_n$ with $\pi(x'')$ between $\pi(x)$ and $\pi(x')$.

So let $x_0 \in T_n$ and consider the bag $B = B_{\pi(x_0)}$ consisting of all vertices $V = (x, y, z)$ in G_n with $\pi(y) \leqslant \pi(x_0) \leqslant \pi(z)$. We partition B as $C_1 \cup C_2 \cup C_3$ where:

(i) A vertex $V = (x, y, z)$ of B belongs to C_1 if $\pi(x) < \pi(x_0)$.
(ii) A vertex $V = (x, y, z)$ of B belongs to C_2 if $\pi(x) > \pi(x_0)$.
(iii) A vertex $V = (x, y, z)$ of B belongs to C_3 if $\pi(x) = \pi(x_0)$. In this case, $x = x_0$.

We now explain why C_1, C_2 and C_3 are independent sets in G_n. This is trivial for C_3. We give the argument for C_1, noting that the argument for C_2 is symmetric.

Suppose that V_1 and V_2 are adjacent vertices in C_1. If the pair (V_1, V_2) determines a strong copy of Q_1, then $\pi(z_1) = \pi(x_2) < \pi(x_0)$, which is a contradiction. On the other hand, if the pair (V_1, V_2) determines a strong copy of Q_2, then $y_1 = x_2$ so that $\pi(y_1) = \pi(x_2) < \pi(x_1) < \pi(x_0)$. Now the geometric series nature of the construction implies that $\pi(z_2) < \pi(x_1) < \pi(x_0)$, which is again a contradiction.

With these observations, we have now proved that path-$\chi(G_n) \leqslant 3$ for all $n \geqslant 1$. This inequality is tight as evidenced by the following five elements of G_n which form a 5-cycle: $V_1 = (\emptyset, 0, 1)$, $V_2 = (1, 10, 11)$, $V_3 = (10, 100, 101)$, $V_4 = (101, 1010, 1011)$ and $V_5 = (1, 101, 11)$. Note that $\pi(101)$ is in $[a_i, b_i]$ for each $i \in [5]$.

Nevertheless, we are able to make a small but important change in the path-decomposition to obtain a decomposition witnessing that path-$\chi(G_n) \leqslant 2$. For the integer n, let $\varepsilon = 2^{-2n}$. Then for each vertex $V = (x, y, z)$ of G_n, we change the interval in the path decomposition for V from $[\pi(y), \pi(z)]$ to $[\pi(y) + \varepsilon, \pi(z) - \varepsilon]$. Our choice of ε guarantees that we still have a path-decomposition of G_n.

Again, we consider an element x_0 of T_n and the bag B consisting of all $V = (x, y, z)$ with $\pi(y)+\varepsilon \leqslant \pi(x_0) \leqslant \pi(z)-\varepsilon$. As before, C_1, C_2 and C_3 are independent sets, although membership in these three sets has been affected by the revised path-decomposition. We claim that $C_1 \cup C_3$ is also an independent set, so that the partition $B = (C_1 \cup C_3) \cup C_2$ witnesses that path-$\chi(G_n) \leqslant 2$.

Suppose to the contrary that $V_1 \in C_1$ and $V_3 \in C_3$ with V_1 adjacent to V_3 in G_n. Clearly, this requires that (V_1, V_3) is associated with a strong copy of the binary tree Q_1 as shown in Fig. 2. This implies that $z_1 = x_0 = x_3$ so that $b_1 = \pi(z_1) - \varepsilon = \pi(x_0) - \varepsilon$, which contradicts the assumption that $a_1 < \pi(x_1) < \pi(x_0) \leqslant b_1$. The contradiction completes the proof of Theorem 3.1.

Acknowledgments

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