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Planar Posets that are Accessible from Below Have Dimension at Most 6

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Abstract

Planar posets can have arbitrarily large dimension. However, a planar poset of height h has dimension at most $192h + 96$, while a planar poset with t minimal elements has dimension at most $2t + 1$. In particular, a planar poset with a unique minimal element has dimension at most 3. In this paper, we extend this result by showing that a planar poset has dimension at most 6 if it has a plane diagram in which every minimal element is accessible from below.

Keywords Dimension · Planar poset · Accessible from below poset

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1 Introduction

A non-empty family $\mathcal{R}$ of linear extensions of a poset P is called a *realizer* of P when $x \leq y$ in P if and only if $x \leq y$ in L for each $L \in \mathcal{R}$. The *dimension* of a poset P, as defined by Dushnik and Miller in their seminal paper [3], is the least positive integer d for which P has a realizer $\mathcal{R}$ with $|\mathcal{R}| = d$.

In recent years, there has been considerable interest in bounding the dimension of a poset in terms of graph theoretic properties of its cover graph and its order diagram. For example, the following papers link the dimension of a poset with tree-width, forbidden minors, sparsity and game coloring numbers: [6, 8, 9, 12, 20].

The results presented here focus on planar posets. Recall that a poset P is said to be *planar* if its order diagram (also called a Hasse diagram) can be drawn without edge crossings in the plane. A construction due to Kelly [10] shows that the standard example S_n is a subposet of a planar poset, so planar posets can have arbitrarily large dimension. However, Joret, Micek and Wiechert [7] proved that any planar poset with height h has dimension at most $192h + 96$. Restricting our attention to planar posets with t minimal elements, Trotter and Wang [18] proved that these posets have dimension at most $2t + 1$. On the other hand, Felsner, Trotter and Wiechert [5] proved that any poset with an outerplanar diagram has dimension at most 4. This is in contrast with the Kelly construction where nearly all minimal elements lie on interior regions of the plane drawing of the diagram. We focus on this distinction, proving a constant bound for the dimension of posets with plane diagrams where every minimal element is accessible from below, which will be carefully defined later in the introduction.

As is well known, a planar poset has an order diagram without edge crossings in which edges are straight line segments. Nevertheless, we elect to consider order diagrams in which covering edges can be piecewise linear, as this convention simplifies our illustrations. Given a planar poset P, a drawing of the order diagram of P using piecewise linear paths for edges such that there are no edge crossings will simply be called a *plane diagram* of P.

In discussing a plane diagram $\mathbb{D}$ for a poset P, we will assume, without loss of generality, that no two points of P lies on the same horizontal or vertical line in the plane. We will also discuss points in the plane which do not correspond to elements of P. In particular, the set of points in the plane which do not correspond to elements of P and do not lie on the piecewise linear covering edges in $\mathbb{D}$ is partitioned into one or more simply connected regions. In general, there can be arbitrarily many bounded regions, however the boundaries of these regions need not be simple closed curves. Among these regions, there is always a unique, unbounded region which is usually referred to as the *exterior region*.

Let $\mathbb{D}$ be a plane diagram for poset P, and let x be a minimal element of P. We will say that x is *accessible from below* when there is a positive number $\epsilon = \epsilon(x)$ so that any point p in the plane which is distinct from x, on the vertical ray emanating downwards from x and within distance ϵ from x is in the exterior region. In turn we say that a plane diagram $\mathbb{D}$ is *accessible from below* if every minimal element of P is accessible from below.

We find it convenient to abbreviate the phrase "accessible from below" with the acronym AFB, so we will say that a minimal element x is AFB in a diagram $\mathbb{D}$, and we will refer to an AFB-diagram. We then say that a poset P is an AFB-poset if it has an AFB-diagram. All AFB-posets are planar, but there are planar posets which are not AFB. Also, an AFB-poset can have many plane diagrams with some of them AFB-diagrams and others not. We illustrate this situation in Fig. 1 where we show two plane diagrams of an AFB-poset P. The diagram on the right is an AFB-diagram while the diagram on the left is not.

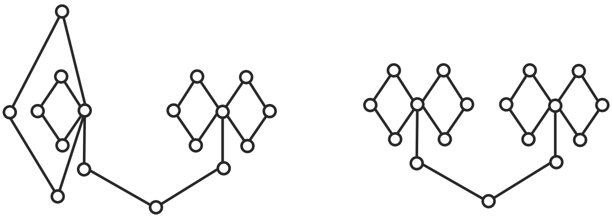

Fig. 1 The diagram on the right is an AFB-Diagram

The principal goal of this paper is to prove the following upper bound on the dimension of an AFB-poset.

Theorem 1 *If P is an* AFB*-poset, then* $\dim(P) \leq 6$.

The remainder of this paper is organized as follows. Some background material necessary for the proof of Theorem 1 is summarized in the next section, and the proof our main theorem is given in Section 3. We close in Section 4 with some comments on the motivation for this line of research and connections with open problems.

We note that for every $d \geq 6$, it is an easy exercise to construct an AFB-poset P for which one cannot argue that $\dim(P) \leq d$ by any of the other known results for planar posets. Although we do not know if our upper bound is best possible, as detailed in Section 4, a finite upper bound on the dimension of AFB-posets is sufficient for our long range goals.

2 Background Material

We use (essentially) the same notation and terminology for working with dimension as has been employed by several authors in recent papers, including: [5, 6, 14, 18, 19], so our treatment will be concise.

Let P be a poset with linear extension L and let $(x, y) \in \mathrm{Inc}(P)$. We say L *reverses* (x, y) when $x > y$ in L. When $S \subset \mathrm{Inc}(P)$, we say that L *reverses* S when L reverses every pair in S. When $\mathcal{R}$ is a family of linear extensions of P, we say $\mathcal{R}$ reverses S when, for each $(x, y) \in S$, there is some $L \in \mathcal{R}$ such that L reverses (x, y). Evidently, the dimension of P is just the minimum size of a non-empty family of linear extensions which reverses $\mathrm{Inc}(P)$.

In the discussion to follow, we sometimes express a linear order on a finite set by writing $[u_1 < u_2 < \cdots < u_r]$, for example.

A subset $S \subset \mathrm{Inc}(P)$ is *reversible* when there is a linear extension L of P which reverses S. When $k \geq 2$, a sequence $\{(x_i, y_i) : 1 \leq i \leq k\}$ of incomparable pairs in P is called an *alternating cycle* (of length k) when $x_i \leq y_{i+1}$ in P for all $i \in [k]$, which should be interpreted *cyclically*, i.e., we also intend that $x_k \leq y_1$ in P. For the balance of the paper, we will use similar cyclic notation without further comment.

An alternating cycle is *strict* when, for each $i \in [k]$, $x_i \leq y_j$ in P if and only if $j = i + 1$ and the sets $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ are k-element antichains.

A poset has dimension 1 if and only if it is a chain. When P is not a chain, the dimension of P is just the least positive integer $d \geq 2$ for which there is a covering $\mathrm{Inc}(P) = S_1 \cup S_2 \cup \cdots \cup S_d$ with S_i reversible for each $i \in [d]$. The following elementary lemma of Trotter and Moore [16] characterizing reversible sets has become an important tool in dimension theory.

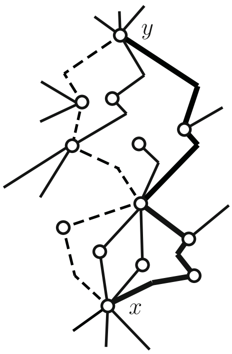

Fig. 2 Left-most and right-most witnessing paths

Lemma 2 *Let P be a poset and let* $S \subseteq \text{Inc}(P)$*. Then the following statements are equivalent.*

(1) *S is reversible.*
(2) *There is no* $k \geq 2$ *for which S contains an alternating cycle of length k.*
(3) *There is no* $k \geq 2$ *for which S contains a strict alternating cycle of length k.*

When $x < y$ in a poset P, we refer to a sequence $W[x, y] = (u_1, u_2, \ldots, u_r)$ of elements of P as a *witnessing path* from x to y when $u_1 = x$, $u_r = y$ and u_i is covered by u_{i+1} in P whenever $1 \leq i < r$. In general, there are many different witnessing paths from x to y and, in most instances, it will not matter which one is chosen.

When $\mathbb{D}$ is a plane diagram for a poset P and $x < y$ in P, we will take advantage of the fact that there is a uniquely determined "left-most" witnessing path from x to y. Analogously, there is a uniquely determined "right-most" witnessing path from x to y. In Fig. 2, we show a portion of a plane diagram where there are a total of 12 witnessing paths from x to y. The left-most path is shown using dotted edges while the right-most path is shown using bold face edges.

We can view a witnessing path $W[x, y]$ as a finite sequence of points of the poset P, but we can also view it as the simply connected (and therefore infinite) set of points in the plane which belong to the covering edges in the path. From the context of the discussion, it should be clear whether we intend a witnessing path to be simply a finite set of points from P or an infinite set of points in the plane. In the same spirit, we will splice witnessing paths together to form simple closed curves in the plane. These will always be infinite sets of points.

At a critical stage in our proof, we will discuss a simple closed curve $\mathcal{E}$ such that the minimal elements of P are on $\mathcal{E}$, while all other elements of P are in the interior of $\mathcal{E}$.

2.1 Planar Posets with a Zero

When a poset has a unique minimal element, that element is usually referred to as a "zero." Dually, if a poset has a unique maximal element, then it is called a "one." We state formally the theorem of Trotter and Moore [16], and give a short synopsis of a more modern proof given in [18], as these details will be important in proving our main theorem.

Theorem 3 *If P is a planar poset and P has a zero, then* $\dim(P) \leq 3$.

Since a poset and its dual have the same dimension, we also know that a planar poset with a one has dimension at most 3.

Given a plane diagram $\mathbb{D}$ for a poset P with a zero, let L_1 be the linear extension of P obtained from a depth-first search using a local left-to-right preference rule. Similarly, let L_2 be another linear extension of P which is also obtained via a depth-first search, but with a right-to-left preference. As noted in [18], for every $(x, y) \in \text{Inc}(P)$, exactly one of the following four statements applies:

(1) x is right of y (i.e. $x > y$ in L_1 and $x < y$ in L_2).
(2) x is left of y (i.e. $x < y$ in L_1 and $x > y$ in L_2).
(3) x is outside y (i.e. $x < y$ in both L_1 and L_2).
(4) x is inside y (i.e. $x > y$ in both L_1 and L_2).

In Fig. 3, we show a plane diagram $\mathbb{D}$ for a poset P with a zero in which 10 is right of 5, 7 is left of 9, 14 is outside 6 and 10 is inside 13.

Accordingly, it is natural to partition $\text{Inc}(P)$ as $\mathcal{R} \cup \mathcal{L} \cup \mathcal{O} \cup \mathcal{I}$, where $\mathcal{R}$ consists of all pairs (x, y) with x right of y, etc. The binary relations $\mathcal{R}$ and $\mathcal{L}$ are complementary in the sense that x is left of y if and only if y is right of x. Similarly, the binary relations $\mathcal{I}$ and $\mathcal{O}$ are complementary. Also both $\mathcal{L}$ and $\mathcal{R}$ are transitive, e.g., if x is left of y and y is left of z, then x is left of z.

The inequality $\dim(P) \leq 3$ is proved by showing that the following sets are reversible: (1) $\mathcal{R} \cup \mathcal{O}$, (2) $\mathcal{L} \cup \mathcal{O}$, and (3) $\mathcal{I}$. The arguments given in [18] for the first two of these statements are constructive, as the desired linear extensions are obtained via depth-first searches. Note that the labeling used in Fig. 3 results from a depth-first search using a local left-to-right preference rule. This linear extension illustrates that $\mathcal{R} \cup \mathcal{O}$ is reversible. A depth-first search using a local right-to-left preference rule will produce a linear extension reversing all pairs in $\mathcal{L} \cup \mathcal{O}$.

To complete the proof, it is then only necessary to show that $\mathcal{I}$ is reversible. However, as pointed out in [18], a somewhat more general result holds: planar posets with t minimal elements have dimension at most $2t + 1$.

Let $\mathbb{D}$ be a plane diagram for a planar poset P (with no restriction on the number of minimal elements of P). When $z \in P$, we let $U_P[z]$ consist of all elements $x \in P$ with $z \leq x$ in P. The subposet $U_P[z]$ is planar and z is a zero. Accordingly, we can classify the incomparable pairs in $U_P[z]$ using the same four labels $\mathcal{R}$, $\mathcal{L}$, $\mathcal{O}$, and $\mathcal{I}$. We will say that

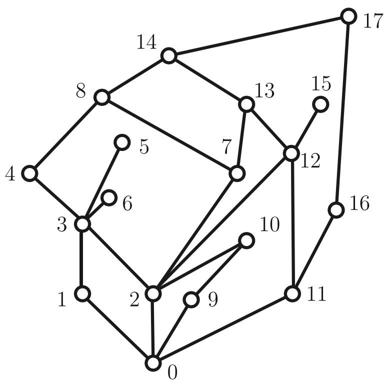

Fig. 3 A planar poset with a zero

an incomparable pair (x, y) in P is an *enclosed* pair when there is some $z \in P$ such that x is inside y in $U_P[z]$.

For the benefit of readers who may be new to arguments using alternating cycles, we give a proof for the following elementary lemma.

Lemma 4 *Let $\mathbb{D}$ be a plane diagram for a poset P. Then the set $\mathcal{S}$ of all enclosed pairs in P is reversible.*

Proof We argue by contradiction, supposing that $\mathcal{S}$ is not reversible. Then by Lemma 2, there is an integer $k \geq 2$ and a strict alternating cycle $\{(x_i, y_i) : 1 \leq i \leq k\}$ of enclosed pairs. For each $i \in [k]$, let z_i be the unique element of P which is highest in the plane with x_i inside y_i in $U_P[z_i]$. Then let y_i' be the unique element of P which is lowest in the plane and satisfies both $y_i' \leq y_i$ in P and x_i is inside y_i' in $U_P[z_i]$. Then there are two witnessing paths $W_1[z_i, y_i']$ and $W_2[z_i, y_i']$ which form a simple closed curve $\mathcal{C}_i$ with x_i in its interior.

Since $x_i \leq y_{i+1}$ in P and $x_i \parallel y_i$ in P, it follows that y_{i+1} is also in the interior of $\mathcal{C}_i$. Therefore y_{i+1} is lower in the plane than y_i'. In particular, y_{i+1}' is lower in the plane than y_i'. This is a contradiction since this statement cannot hold for all $i \in [k]$. □

In a dual manner, when $\mathbb{D}$ is a plane diagram for a poset P and $z \in P$, we define $D_P[z]$ as the subposet consisting of all $x \in P$ with $x \leq z$ in P. The subposet $D_P[z]$ is planar, and the element z is a one. Now we can classify the incomparable pairs in the subposet $D_P[z]$ using the same four labels but applied with the obvious dual interpretation. In general, if (x, y) is an incomparable pair in a subposet of the form $D_P[z]$, then any of the four labels may be correct for the pair (x, y). However, if $\mathbb{D}$ is an AFB-diagram for a poset P, then two of the four labels cannot be applicable. We state formally the following nearly self-evident proposition for emphasis. It does not hold for planar posets in general.

Proposition 5 *Let $\mathbb{D}$ be an* AFB*-diagram for a poset P, and let $z \in P$. If (x, y) is an incomparable pair in $D_P[z]$, then either x is left of y in $D_P[z]$ or x is right of y in $D_P[z]$. Furthermore, if $z' \in P$ and $x, y \in D_P[z']$, then x is left of y in $D_P[z]$ if and only if x is left of y in $D_P[z']$.*

3 Proof of our Main Theorem

In this section, we prove Theorem 1, i.e., we show that if P is an AFB-poset, then $\dim(P) \leq 6$. Our first step is to reduce the problem to a somewhat simpler one.

Reduction To show that the dimension of any AFB-poset is at most d, it suffices to show that whenever $\mathbb{D}$ is an AFB-diagram for a poset P, the set of all incomparable pairs in $\mathrm{Min}(P) \times P$ can be covered by $d - 1$ reversible sets.

Proof Let P be an AFB-poset. To show that $\dim(P) \leq d$, we need to show that there is a covering of the set of all incomparable pairs of P by d reversible sets. Let $\mathbb{D}$ be an AFB-diagram for P. We will now show that $\mathbb{D}$ can be modified into an AFB-diagram $\mathbb{D}'$ for a poset P' such that:

(1) P' contains P as a subposet.
(2) If (x, y) is an enclosed pair in P, then (x, y) is an enclosed pair in P'.

(3) If (x, y) is an incomparable pair of P and is not an enclosed pair in P, then there is a minimal element $x' \in P'$ with $x' \leq x$ in P' and $x' \parallel y$ in P'.

We will then let S_0 consist of all incomparable pairs (x, y) in P such that (x, y) is an enclosed pair in P'. The set S_0 is reversible by Lemma 4, and it contains all enclosed pairs in P. It remains to consider the incomparable pairs in P which are not enclosed in P'. In view of the third condition for P', if the incomparable pairs $(x', y) \in \text{Min}(P') \times P'$ in P' can be covered by $d - 1$ reversible sets, it follows that the set of all incomparable pairs of P can be covered by d reversible sets. So it only remains to explain how the poset P' should be constructed from P.

Let S be the set of all incomparable pairs of P which are not enclosed pairs in P and do not belong to $\text{Min}(P) \times P$. If $S = \emptyset$, simply take $\mathbb{D}' = \mathbb{D}$ and $P' = P$. So we may assume that $S \neq \emptyset$. Let $r = |S|$ and let $S = \{(x_i, y_i) : 1 \leq i \leq r\}$ be an arbitrary labeling of the pairs in S.

To initialize a recursive construction, we set $\mathbb{D}_0 = \mathbb{D}$ and $P_0 = P$. We will now explain how to construct a sequence $\{(\mathbb{D}_i, P_i) : 1 \leq i \leq r\}$ such that for each $i \in [r]$, $\mathbb{D}_i$ is an AFB-diagram for the poset P_i where $M_i = \text{Min}(P_i)$. The construction will ensure that P_{i-1} is a subposet of P_i and M_{i-1} is a subset of M_i whenever $1 \leq i \leq r$. Furthermore, for each $1 \leq j \leq i \leq r$, either (x_j, y_j) is an enclosed pair in P_i or there is a minimal element $x'_j \in M_i$ such that $x'_j \leq x_j$ in P_i and $x'_j \parallel y_j$ in P_i. The AFB-poset P' is just P_r.

Now suppose that $0 \leq i < r$ and that we have defined the AFB-diagram $\mathbb{D}_i$ for P_i. We then consider the pair $(x, y) = (x_{i+1}, y_{i+1})$. Let x' be the uniquely determined element of P which is lowest point in the plane and satisfies $x' \leq x$ in P_i and $x' \parallel y$ in P_i. If (x', y) is an enclosed pair in P_i, so is (x, y). Accordingly, if $x' \in \text{Min}(P_i)$, or (x', y) is an enclosed pair, we simply take $\mathbb{D}_{i+1} = \mathbb{D}_i$ and $P_{i+1} = P_i$.

Now suppose (x', y) is not an enclosed pair in P_i, and x' is not a minimal element in P_i. It follows that x' covers one or more elements in P_i. We claim that x' has a unique lower cover. Suppose to the contrary that x' covers distinct elements u and u' in P_i. In view of our choice of x', we know $u, u' \in D_{P_i}[y]$. However, since (x', y) is not an enclosed pair in P_i and $u \parallel u'$, the induced AFB-diagram of the subposet of P_i determined by $\{u, u', x', y\}$ must be drawn so that u is left of u' in either $D_{P_i}[x']$ or $D_{P_i}[y]$ while u is right of u' in the other which violates Proposition 5. The contradiction shows that x' covers a unique point u as claimed.

Our choice of x' implies that $u < y$ in P_i. We consider the first edge of a witnessing path $W[u, y]$ and the edge ux'. The construction for $\mathbb{D}_{i+1}$ depends on which of these two edges is left of the other at u. In Fig. 4, we show the first edge of $W[u, y]$ on the left, so the following discussion will be reversed if the edge ux is on the left.

Starting with u and traveling down in the diagram, we always proceed to the right-most lower cover until we reach a minimal element of P_i. In Fig. 4, we suggest that this would result in the chain $(u > v > z > w)$ and it should be clear how the following details should be modified if the actual chain is of a different length.

Starting just above u and heading downward, we insert new points very close to the existing vertices—together with intermediate vertices to ensure that the resulting figure is a diagram. This results in a new minimal element x'' with $x'' < x' \leq x$ in P_{i+1} and $x'' \parallel y$ in P_{i+1}. Again, we refer to Fig. 4 as an example of how these changes are to be made. Note that no new comparabilities are introduced among the points of P_{i+1} with the addition of these new points.

With this construction in hand, the proof for the reduction is complete. □

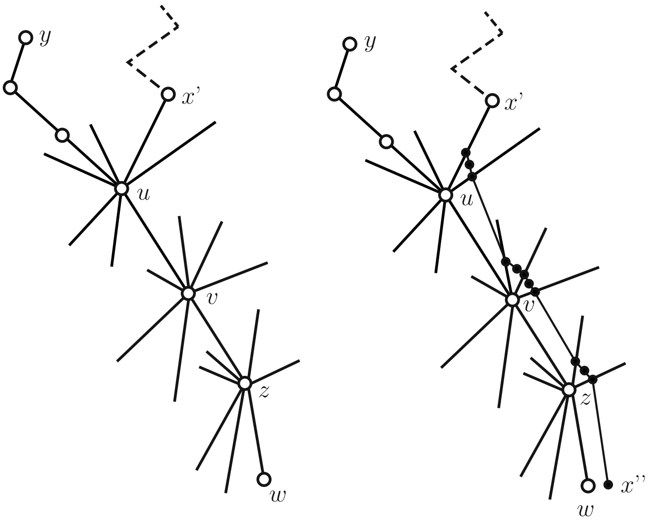

Fig. 4 The construction for the reduction

Given an AFB-diagram for a poset P, we know of no simple argument to show that the set of incomparable pairs from $\text{Min}(P) \times P$ can be covered with a bounded number of reversible sets, but in time, we will show that 5 are enough. With the reduction, this completes the proof that $\dim(P) \leq 6$ when P is an AFB-poset. However, to simplify the proof, we will first prove a weaker result asserting that the set of incomparable pairs from $\text{Min}(P) \times P$ can be covered by 7 reversible sets. The slight modification necessary to lower 7 to 5 will be presented later.

Clearly, it is enough to prove our bound of 7 when P is connected and has at least two minimal elements. We let $\mathcal{S}_0$ denote the set of all incomparable pairs in $\text{Min}(P) \times P$, and we abbreviate the set $\text{Min}(P)$ as M.

Since $\mathbb{D}$ is an AFB-diagram, it is easy to see that there is a simple closed curve $\mathcal{E}$ in the plane satisfying the following requirements:

(1) All elements of M are on $\mathcal{E}$.
(2) All elements of $P - M$ are in the interior of $\mathcal{E}$.
(3) If x covered by y in P, then all points of the plane which are on the covering edge from x to y in the diagram are in the interior of $\mathcal{E}$, except x when $x \in M$.

We illustrate such a curve in Fig. 5 where we show $\mathcal{E}$ using dashed lines. We find it natural to refer to $\mathcal{E}$ as an *envelope for* $\mathbb{D}$.

Starting at an arbitrary minimal element m_1, we label the elements of M as they appear in a counter-clockwise traversal of $\mathcal{E}$ to obtain a linear order

$$L = [m_1 < m_2 < \cdots < m_n]$$

on M. For each element $y \in P$, we let $M[y] = M \cap D_P[y]$.

We will make repeated use of the following elementary proposition. In fact, a stronger result holds, but this is the exact form we need.

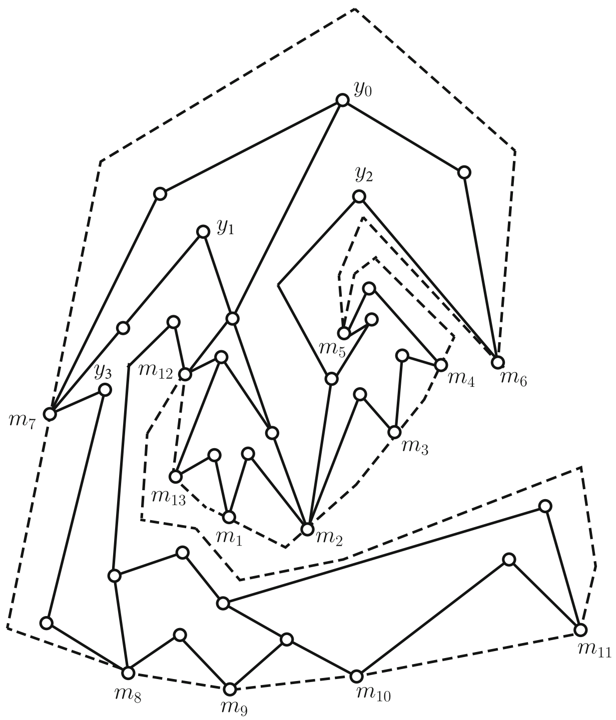

Fig. 5 An envelope for an AFB-poset

Proposition 6 *Let $y \in P$, and let m and m' be distinct elements of M with $m, m' \in D_P[y]$. Then let Z be the subposet of P consisting of all elements $z \in P$ with $z \in D_P[y]$ such that $m, m' \in D_P[z]$. Then the subposet Z has a unique minimal element which we will denote $z(y, m, m')$.*

We will also make repeated use of a construction that produces simple closed curves and regions in the plane. Again, let $y \in P$ and let (m, m') be an *ordered* pair of distinct elements of $M[y]$. Form a path $\mathcal{E}[m, m']$ by traversing the simple closed curve $\mathcal{E}$ in a counter-clockwise direction starting at m and stopping at m'. Now $\mathcal{E}[m, m']$ and $\mathcal{E}[m', m]$ share only m and m' as endpoints. Their union is the entire curve $\mathcal{E}$.

Let $z = z(y, m, m')$. We then take witnessing paths from m and m' to z using the following convention: If m is left of m' in $D_P[z]$, then we take $W[m, z]$ as the right-most witnessing path from m to z, while we take $W[m', z]$ as the left-most path from m' to z. These conventions are reversed if m' is left of m in $D_P[z]$.

In either situation, the two witnessing paths $W[m, z]$ and $W[m', z]$ together with the path $\mathcal{E}[m, m']$ form a simple closed curve which we denote $\mathcal{C}(y, m, m')$. Also, we let $\mathcal{R}(y, m, m')$ denote the region in the plane enclosed by $\mathcal{C}(y, m, m')$. Note that y is on $\mathcal{C}(y, m, m')$ when $y = z(y, m, m')$. However, when $y \neq z(y, m, m')$, y is in the exterior of $\mathcal{R}(y, m, m')$ when m is left of m' in $D_P[y]$, and y is in the interior of $\mathcal{R}(y, m, m')$ when m is right of m' in $D_P[y]$.

Now we proceed to state our main lemma.

Lemma 7 *Let $\mathbb{D}$ be an* AFB*-diagram for a poset P. Then the set of all incomparable pairs of P in* $\mathrm{Min}(P) \times P$ *can be covered by 7 reversible sets.*

To prove this, we will use the linear order L associated with $\mathcal{E}$ to label the incomparable pairs in $\mathcal{S}_0$, the set of incomparable pairs of P in $M \times P$ using the following 8 labels:

$$1A \quad 1B \quad 1C \quad 2A \quad 2B \quad 2C \quad 2D \quad 2E.$$

The integer part of the label applied to a pair (x, y) depends only on y while the letter in the label depends on both x and y.

Let y be an element of P. Then the elements of $M[y]$ are linearly ordered from left-to-right in $D_P[y]$. We let $s(y)$ and $t(y)$ denote, respectively, the least element and the greatest element of $M[y]$ in this linear order. Let $|M[y]| = r$ and let $[u_1 < u_2 < \cdots < u_r]$ be the left-to-right order on $M[y]$ in $D_P[y]$, so that $s(y) = u_1$ and $t(y) = u_r$.

However, the elements of $M[y]$ are also linearly ordered in L. Now we let $a(y)$ and $b(y)$ denote, respectively, the least element and the greatest element of $M[y]$ in L. Since the envelope $\mathcal{E}$ is traversed in a counter-clockwise manner, it is easy to see that y can be characterized as one of two types, since exactly one of the following two statements holds for y:

Type 1. $u_1 < u_2 < \cdots < u_r$ in L.
Type 2. There is an integer j with $1 < j \leq r$ such that:
$u_j < u_{j+1} < \cdots < u_r < u_1 < u_2 < \cdots < u_{j-1}$ in L.

We note that an element $y \in P$ is Type 1 when $|M[y]| = 1$. In general, when y is Type 1, $a(y) = s(y) \leq t(y) = b(y)$ in L. When y is Type 2, $a(y) \leq t(y) < s(y) \leq b(y)$ in L. Also, we observe that either $a(y) = b(y)$ or $a(y)$ is left of $b(y)$ in $D_P[y]$ when y is Type 1. However, $a(y)$ is right of $b(y)$ in $D_P[y]$ when y is Type 2.

Now let (x, y) be a pair in $\mathcal{S}_0$. If y is Type 1, we will say that (x, y) is Type $1A$ if $x < a(y)$ in L; Type $1B$ if $a(y) < x < b(y)$ in L; and Type $1C$ if $x > b(y)$ in L. In Fig. 5, the elements y_2 and y_3 are Type 1. The pairs (m_1, y_2) and (m_5, y_3) are Type $1A$; the pairs (m_3, y_2) and (m_5, y_2) are Type $1B$; and the pairs (m_8, y_2) and (m_{12}, y_3) are Type $1C$.

When y is Type 2, we say the pair (x, y) is Type $2A$ if $x < a(y)$ in L; Type $2B$ if $a(y) < x < t(y)$ in L; Type $2C$ if $t(y) < x < s(y)$ in L; Type $2D$ if $s(y) < x < b(y)$ in L; and Type $2E$ if $x > b(y)$ in L. In Fig. 5, the elements y_0 and y_1 are Type 2. Now (m_1, y_0) and (m_1, y_1) are Type $2A$; (m_4, y_0) is Type $2B$; (m_6, y_1) is Type $2C$; (m_8, y_0) and (m_{10}, y_1) are Type $2D$; and (m_{13}, y_0) and (m_{13}, y_1) are Type $2E$.

We then define a covering of $\mathcal{S}_0$ by six sets defined as follows:

(1) $\mathcal{S}_1$ consists of all Type $1A$ and $2A$ pairs.
(2) $\mathcal{S}_2$ consists of all Type $1C$ and $2E$ pairs.
(3) $\mathcal{S}_3$ consists of all Type $1B$ pairs.
(4) $\mathcal{S}_4$ consists of all Type $2B$ pairs.
(5) $\mathcal{S}_5$ consists of all Type $2D$ pairs.
(6) $\mathcal{S}_6$ consists of all Type $2C$ pairs.

We pause to examine the AFB-poset shown in Fig. 6 just to understand that there are obstacles to overcome in covering $\mathcal{S}_0$ by a small number of reversible sets. Referring to Fig. 6, the set $\mathcal{S}_1 \cup \mathcal{S}_2$ need not be reversible since (x_4, y_4) is Type $1A$ and (x_5, y_5) is Type $1C$, but together these form a strict alternating cycle. Also, (x_1, y_1) and (x_2, y_2) are Type $2C$ while (x_3, y_3) is Type $2B$. No reversible set can contain any two of these three pairs so $S_4 \cup S_6$ is not reversible and neither is S_6.

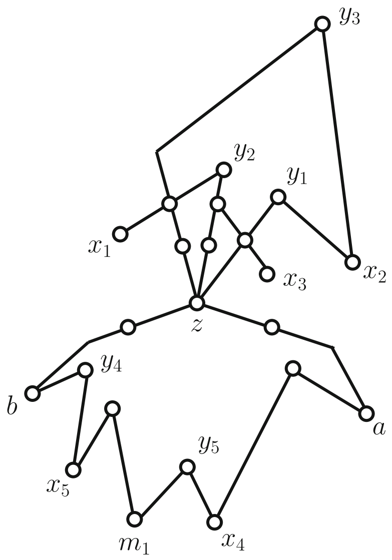

Fig. 6 Challenges in reversing pairs in $\mathcal{S}_0$

Despite these challenges, the proof of Lemma 7 and the proof of the (weak) upper bound $\dim(P) \leq 8$ will be complete once we have verified the following claim.

Claim 2 Each of the sets in the family $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5\}$ is reversible. Furthermore, the set $\mathcal{S}_6$ can be covered by two reversible sets.

Proof We will examine one set at a time, grouping sets with symmetric arguments.

Case $\mathcal{S}_1$ ($\mathcal{S}_2$). We will first give a proof by contradiction to show that $\mathcal{S}_1$ is reversible. The argument for $\mathcal{S}_2$ is symmetric. Suppose to the contrary that $\mathcal{S}_1$ is not reversible. Let $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ be a strict alternating cycle contained in $\mathcal{S}_1$. For each $i \in [k]$, let $a_i = a(y_i)$, the least element of $M[y_i]$ in the linear order L.

For each $i \in [k]$, since $(x_i, y_i) \in \mathcal{S}_1$, we know that $x_i < a_i$ in L. On the other hand, we know that $x_i \leq y_{i+1}$ in P. Therefore $a_{i+1} \leq x_i$ in L. In turn, this implies $a_{i+1} < a_i$ in L. Clearly, this statement cannot hold for all $i \in [k]$. The contradiction completes the proof for this part of the claim.

Case $\mathcal{S}_3$. Now we give a proof by contradiction to show that the set $\mathcal{S}_3$ of Type $1B$ pairs is reversible. This argument will be more substantive than the preceding case. Suppose that $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ is a strict alternating cycle of pairs from $\mathcal{S}_3$. For each $i \in [k]$, we let $a_i = a(y_i)$, $b_i = b(y_i)$, $s_i = s(y_i)$ and $t_i = t(y_i)$. Since $(x_i, y_i) \in \mathcal{S}_3$, we know

$$a_i = s_i < x_i < t_i = b_i \quad \text{in } L.$$

Furthermore, we know a_i is left of b_i in $D_P[y_i]$. Let $z_i = z(y_i, a_i, b_i)$, $\mathcal{E}_i = \mathcal{E}[a_i, b_i]$, $\mathcal{C}_i = \mathcal{C}(y_i, a_i, b_i)$ and $\mathcal{R}_i = \mathcal{R}(y_i, a_i, b_i)$

Now let $i \in [k]$ be arbitrary. Since $a_i < x_i < b_i$ in L, we know x_i is on the path $\mathcal{E}_i$. Since S is a strict alternating cycle, we know $x_i \leq y_{i+1}$ in P and $y_i \parallel y_{i+1}$ in P. Let $W[x_i, y_{i+1}]$ be an arbitrary witnessing path. Clearly, y_{i+1} is not a minimal element in P since (x_{i+1}, y_{i+1}) has Type 1B, so all points in the plane on the witnessing path $W[x_i, y_{i+1}]$ *except* x_i are in the interior of $\mathcal{C}_i$.

Next we consider the set $M[y_{i+1}]$ which includes x_i. We assert that all elements of $M[y_{i+1}]$ are on $\mathcal{E}_i$. To see this, suppose $u \in M[y_{i+1}]$, and u is not on $\mathcal{E}_i$. Then u is in the exterior of $\mathcal{C}_i$. Let $W[u, y_{i+1}]$ be an arbitrary witnessing path. Then this path must intersect the boundary of $\mathcal{C}_i$, and this forces $u < y_i$ in P which disagrees with the choice of a_i and b_i. The contradiction confirms our assertion.

We conclude that

$$a_i \leq a_{i+1} \quad \text{and} \quad b_{i+1} \leq b_i \quad \text{in } L. \tag{1}$$

Of course, we also know that $a_{i+1} \leq b_i$ in L, but we elect to write the two inequalities in Eq. 1 in a weak form. Since $i \in [k]$ was arbitrary, these inequalities hold for all $i \in [k]$. We conclude that there are minimal elements a_0 and b_0 so that $a_i = a_0$ and $b_i = b_0$ for each $i \in [k]$. The rules for determining z_i and the witnessing paths $W[a_0, z_i]$ and $W[b_0, z_i]$ force $\mathcal{R}_{i+1}$ to be a *proper* subset of $\mathcal{R}_i$. Clearly, this is a contradiction since the strict set inclusion statement cannot hold for all $i \in [k]$. This completes the proof that the set $\mathcal{S}_3$ consisting of all Type $1B$ pairs is reversible.

Case $\mathcal{S}_4$ ($\mathcal{S}_5$). Next, we argue by contradiction that the set $\mathcal{S}_4$ of all Type $2B$ pairs is reversible. The argument for the set $\mathcal{S}_5$ of all Type $2D$ pairs is symmetric. Suppose to the contrary that $\mathcal{S}_4$ is not reversible, and let $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ be a strict alternating cycle contained in $\mathcal{S}_4$. We use the same abbreviations as in the preceding case for a_i, b_i, s_i and t_i. Since (x_i, y_i) is Type $2B$, we know $a_i < x_i < t_i < s_i \leq b_i$ in L.

For each $i \in [k]$, we set $z_i = z(y_i, a_i, t_i)$, $\mathcal{E}_i = \mathcal{E}[a_i, t_i]$, $\mathcal{C}_i = \mathcal{C}(y_i, a_i, t_i)$ and $\mathcal{R}_i = \mathcal{R}(y_i, a_i, t_i)$. It follows that y_{i+1} is in the interior of $\mathcal{R}_i$. We now assert that all points of $M[y_{i+1}]$ come from $\mathcal{E}[a_i, b_i]$. To see this, let u be an element of $M[y_{i+1}]$ which does not belong to $\mathcal{E}(a_i, b_i)$. Then u is in the exterior of $\mathcal{R}_i$, and a witnessing path $W[u, y_{i+1}]$ would have to intersect $\mathcal{C}_i$. This forces $u < y_i$ in P so that $u \in \mathcal{E}(a_i, b_i)$, as desired. In turn, this implies that inequality Eq. 1 holds. Since this inequality holds for all $i \in [k]$, we know there are elements $a_0, b_0 \in M$ so that $a_i = a_0$ and $b_i = b_0$ for all $i \in [k]$.

Now we assert that $t_{i+1} \leq t_i$ in L for all $i \in [k]$. Because (x_{i+1}, y_{i+1}) is in S and has Type 2B, $a_0 < t_{i+1}$ in the left-to-right order. Let $W[t_{i+1}, y_{i+1}]$ be any witnessing path. Then this path intersects $\mathcal{C}_i$. Let v be the unique element of P which is lowest in the plane, and is common to $W[t_{i+1}, y_{i+1}]$ and the boundary of $\mathcal{C}_i$. Clearly, $v < z_i$ in P so $t_{i+1} \in D_P[y_i]$. Therefore, in the left-to-right order of $M[y_i]$, $t_{i+1} \leq t_i$ by the definition of t_i. Since $a_0 < t_{i+1} \leq t_i$ in the left-to-right order, then $t_{i+1} \leq t_i$ in L. Since S is a strict alternating cycle, we know that there is a point $t_0 \in M$ so that $t_i = t_0$ for all $i \in [k]$.

Now the same argument used in proving that the set $\mathcal{S}_3$ of all Type $1B$ pairs is reversible shows that region $\mathcal{R}_{i+1}$ is a proper subset of $\mathcal{R}_i$. Clearly, this statement cannot hold for all $i \in [k]$, and this completes the proof that the set $\mathcal{S}_4$ consisting of all Type $2B$ pairs is reversible.

Case $\mathcal{S}_6$. Now we turn to the last statement of Claim 2 where we must prove that the set $\mathcal{S}_6$ of all Type $2C$ pairs can be covered by two reversible sets. Note that two Type $2C$ pairs in Fig. 6 show that $\mathcal{S}_6$ may not be reversible.

Let (x, y) be a Type $2C$ pair, and let $a = a(y)$, $b = b(y)$ and $z = z(y, a, b)$. We will say that (x, y) is *left-biased* if there is a Type 2 element $y' \in P$ such that $(1')$ $a(y') = a$ and $b(y') = b$; $(2')$ $z(y', a, b) = z$; and $(3')$ x is left of b in $D_P[y']$. Similarly, we will say that (x, y) is *right-biased* if there is an a Type 2 element y'' satisfying $(1'')$ $a(y'') = a$ and $b(y'') = b$; $(2'')$ $z(y'', a, b) = z$; and also $(3'')$ x is right of a in $D_P[y'']$.

We assert that there is no Type $2C$ pair (x, y) which is both left-biased and right-biased. If this were to happen, we observe that a, b, z belong to both $D_P[y']$ and $D_P[y'']$. We would require that $x < a < b$ in the left-to-right order on $D_P[y']$ and $a < b < x$ in the left-to-right order in $D_P[y'']$. In particular, both the pairs (a, x) and (b, x) violate Proposition 5. This proves that the assertion is correct.

We now show that the set $\mathcal{S}'$ of all Type $2C$ pairs which are *not* right-biased is reversible. A symmetric argument shows that the set $\mathcal{S}''$ of all Type $2C$ pairs which are *not* left-biased is reversible. Once this has been accomplished, the proof that the set $\mathcal{S}_6$ consisting of all Type $2C$ pairs can be covered by two reversible sets will be complete.

We argue by contradiction and let $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ be a strict alternating cycle of Type $2C$ pairs, none of which are left-biased. For each $i \in [k]$, we use the now standard abbreviations a_i, b_i, s_i, t_i. We then take $z_i = z(y_i, a_i, b_i)$, $\mathcal{E}_i = \mathcal{E}[a_i, b_i]$, $\mathcal{C}_i = \mathcal{C}(y_i, a_i, b_i)$ and $\mathcal{R}_i = \mathcal{R}(y_i, a_i, b_i)$.

Arguments just like those applied earlier show that inequality Eq. 1 holds. Therefore, there are elements $a_0, b_0 \in M$ such that $a_i = a_0$ and $b_i = b_0$ for all $i \in [k]$.

Now let $i \in [k]$ be arbitrary. We then observe that $\mathcal{R}_{i+1}$ is a proper subset of $\mathcal{R}_i$ *unless* $z_{i+1} = z_i$. In this case, $\mathcal{R}_{i+1} = \mathcal{R}_i$. It follows that there is an element $z_0 \in P$ and a simple closed curve $\mathcal{C}_0$ enclosing a region $\mathcal{R}_0$ so that $z_i = z_0$, $\mathcal{C}_i = \mathcal{C}_0$ and $\mathcal{R}_i = \mathcal{R}_0$ for all $i \in [k]$.

After a relabeling if necessary, we may assume that $s_1 \leq s_i$ in L for each $i \in [k]$. Then $t_1 < x_1 < s_1$ in L. Since $x_1 < y_2$ in P, either $s_2 \leq x_1 < b_0$ in L or $a_0 < x_1 \leq t_2$ in L. If $s_2 \leq x_1 < b_0$ in L, then $s_2 < s_1$ in L which is false. We conclude that $a_0 < x_1 \leq t_2$ in L. Therefore, x_1 is right of a_0 in $D_P[y_2]$. This shows that (x_1, y_1) is right-biased. The contradiction completes the proof. □

As promised, we now show how to improve Claim 2 by showing the set $\mathcal{S}_0$ can be covered by 5 reversible sets. This will be accomplished by proving the next two claims.

Claim 3 The set $\mathcal{S}_3 \cup \mathcal{S}_4$ of all pairs which are either Type $2B$ or Type $2D$ is reversible.

Proof We first show by contradiction that $\mathcal{S}_3 \cup \mathcal{S}_4$ is reversible. Let $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ be a strict alternating cycle of pairs from $\mathcal{S}_3 \cup \mathcal{S}_4$ In view of our earlier arguments, there must be at least one pair in S of Type $2B$ and at least one pair of Type $2D$.

The abbreviations a_i, b_i, s_i, t_i are just as before. Now we know that $a_i \leq t_i < s_i \leq b_i$ in L. Furthermore, if (x_i, y_i) is Type $2B$, we know $a_i < x_i < t_i$ in L, and if (x_i, y_i) is Type $2D$, we know $s_i < x_i < b_i$ in L.

Now let $i \in [k]$. If $a_i = t_i$, we set $z_i = a_i$, and we let $\mathcal{R}_i$ be the region in the plane consisting only of the point a_i. If $a_i < t_i$ in L, we set $z_i = z(y_i, a_i, t_i)$, $\mathcal{C}_i = \mathcal{C}(y_i, a_i, b_i)$ and $\mathcal{R}_i = \mathcal{R}(y_i, a_i, t_i)$. Analogously, if $s_i = b_i$, we set $v_i = b_i$ and we take $\mathcal{T}_i$ as the region in the plane consisting only of the point b_i. If $s_i < b_i$ in L, we let $v_i = z(y_i, s_i, b_i)$, $\mathcal{D}_i = \mathcal{C}(y_i, s_i, b_i)$ and $\mathcal{T}_i = \mathcal{R}(y_i, s_i, b_i)$,

Repeating arguments already presented, we quickly learn that there are elements $a_0, b_0, s_0, t_0 \in M$ and elements $z_0, v_0 \in P$ so that $a_i = a_0$, $b_i = b_0$, $z_i = z_0$, and $v_i = v_0$ for all $i \in [k]$.

If $i \in [k]$ and (x_i, y_i) is Type $2B$, then it is easy to see that $\mathcal{R}_{i+1} \subsetneq \mathcal{R}_i$ while $\mathcal{T}_{i+1} = \mathcal{T}_i$. Analogously, if (x_i, y_i) is Type $2D$ then $\mathcal{T}_{i+1} \subsetneq \mathcal{T}_i$ while $\mathcal{R}_{i+1} = \mathcal{R}_i$. Clearly, these statements result in a contradiction, so we have completed the proof that $\mathcal{S}_3 \cup \mathcal{S}_4$ is reversible. □

Claim 4 The set $\mathcal{S}_7$ consisting of all pairs which are either Type $1B$ or Type $2C$ but not right-biased is reversible.

Proof Now we prove by contradiction that $\mathcal{S}_7$, which consists of all Type $1B$ pairs and all Type $2C$ pairs which are not right-biased is reversible. Let $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ be a strict alternating cycle contained in $\mathcal{S}_7$. Then we know that S contains both a Type $1B$ pair and a Type $2C$ pair.

Now suppose that $i \in [k]$. We set $a_i = a(y_i)$, $b_i = b(y_i)$, $z_i = z(y_i, a_i, b_i)$, $\mathcal{C}_i = \mathcal{C}(y_i, a_i, b_i)$ and $\mathcal{R}_i = \mathcal{R}(y_i, a_i, b_i)$. Then y_{i+1} is in the interior of $\mathcal{R}_i$ and all elements of $M[y_{i+1}]$ are on the path $\mathcal{E}(a_i, b_i)$. It follows that the inequalities in Eq. 1 hold. We conclude that there are elements $a_0, b_0 \in M$ such that $a_i = a_0$ and $b_i = b_0$ for all $i \in [k]$.

Let i and j be integers in $[k]$ so that (x_i, y_i) is Type $1B$ and (x_j, y_j) is Type $2C$. Then a_0 is left of b_0 in $D_P[y_i]$ and a_0 is right of b_0 in $D_P[y_j]$. These statements contradict Proposition 5. With these observations, the proof of Claim 3 is complete. This also completes the proof of our main theorem. □

4 Closing Comments and Open Problems

We pause to explain our motivation in studying the class of AFB-posets. Let P be a planar poset and let x_0 be an arbitrary minimal element of P. Then set $A_0 = \{x_0\}$ and let B_0 consist of all elements y in P such that $y > x_0$ in P. If $i \geq 0$ and we have defined a sequence $(A_0, B_0, A_1, B_1, \ldots, A_i, B_i)$ of pairwise disjoint subposets of P and their union is a proper connected subposet Q of P, we let A_{i+1} consist of all elements $x \in P - Q$ for which there is some $y \in B_i$ such that $x < y$ in P. Also, when $Q \cup A_{i+1}$ is a proper subposet of P, we take B_{i+1} as the set of all $y \in P - (Q \cup A_{i+1})$ for which there is some $x \in A_{i+1}$ for which $x < y$ in P.

The resulting partition of P is now known as an *unfolding* of P, and this concept has been used in several papers, including [7, 12, 14]. The key feature for our purposes is that for all $i \geq 0$, the subposet B_i is an AFB-poset, and the dual of the subposet A_i is an AFB-poset. As is well known there is some $i \geq 0$ for which:

$$\max\{\dim(A_i \cup B_i), \dim(B_i \cup A_{i+1})\} \geq \dim(P)/2.$$

It follows that when the dimension of P is *very* large, we now know that there is a subposet of P in which the difficulty of the dimension problem has a "bipartite vor," i.e., the poset is the union of two relatively simple subposets, one a down set and the other an up set and the challenge is to reverse incomparable pairs of the form (x, y) fla where x is in the down set and y is in the up set. Reversing the remaining incomparable pairs takes at most 6 linear extensions.

Our motivation for this line of research has from the outset been to develop machinery for attacking the following long-standing and apparently quite challenging conjectures:

Conjecture 8 A planar poset with large dimension contains a large standard example, i.e., for every $d \geq 2$, there exists a constant d_0 so that if P is a planar poset and $\dim(P) \geq d_0$, then P contains the standard example S_d as a subposet.

We believe, but cannot be certain, that the first reference to this conjecture is on page 119 in [15], as it has become part of the folklore of the subject.

In fact, probably the following considerably stronger conjecture is true.

Conjecture 9 For every pair (n, d) of positive integers with $d \geq 2$, there is an integer d_0 so that if P is a poset and $\dim(P) \geq d_0$, then either P contains the standard example S_d or the cover graph of P contains a K_n minor.

In just the last two years, there has been considerable interest in two variants of the original Dushnik-Miller notion of dimension. They are called *Boolean dimension* and *local dimension*. We refer readers to [1, 2, 4, 11, 17] for definitions and results.

Specific to our interests here is the proof by Bosek, Grytczuk and Trotter [2] that local dimension is not bounded for planar posets. The following conjecture is due to Nešetřil and Pudlak and is given in question form in their 1989 paper [13] in which the concept of Boolean dimension is first introduced.

Conjecture 10 The Boolean dimension of planar posets is bounded, i.e., there is a constant d_0 so that if P is a planar poset, then the Boolean dimension of P is at most d_0.

We believe that the results presented here will prove useful in attacking this conjecture with the assistance of the concept of unfolding.

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