

CONCEPTS OF DIMENSION FOR CONVEX GEOMETRIES

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ABSTRACT. Let X be a finite set. A family P of subsets of X is called a *convex geometry* with ground set X if (1) $\emptyset, X \in P$; (2) $A \cap B \in P$ whenever $A, B \in P$; and (3) if $A \in P$ and $A \neq X$, there is an element $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$. As a non-empty family of sets, a convex geometry has a well defined VC-dimension. In the literature, a second parameter, called convex dimension, has been defined expressly for these structures. Partially ordered by inclusion, a convex geometry is also a poset, and four additional dimension parameters have been defined for this larger class, called Dushnik-Miller dimension, Boolean dimension, local dimension, and fractional dimension, respectively. For each pair of these six dimension parameters, we investigate whether there is an infinite class of convex geometries on which one parameter is bounded and the other is not.

1. STATEMENT OF RESULTS

The primary goal of this paper is to investigate concepts of dimension for a special class of posets called convex geometries. These concepts are called convex dimension, VC-dimension, Dushnik-Miller dimension, Boolean dimension, fractional dimension, and local dimension, denoted $\text{cdim}(P)$, vdim , $\text{dim}(P)$, $\text{bdim}(P)$, $\text{ldim}(P)$, and $\text{fdim}(P)$, respectively. We will also study a related poset parameter, called standard example number, denoted $\text{se}(P)$. When P is a convex geometry, the following inequalities are known to hold: (1) $\text{cdim}(P) \geq \text{dim}(P) \geq \max\{\text{vdim}(P), \text{bdim}(P), \text{fdim}(P), \text{ldim}(P), \text{se}(P)\}$; and (2) $\text{se}(P) \geq \text{vdim}(P)$ unless $\text{vdim}(P) = 2$ and $\text{se}(P) = 1$. For readers who are familiar with concepts of dimension *and* convex geometries, we state here in the results of this paper. Motivation, definitions, and essential preliminary material will be provided in Sections 2 and 3. Proofs are given in Sections 4 through 8, and we close with some comments on open problems that remain in Section 9.

Our first result separates Dushnik-Miller dimension and convex dimension.

Theorem 1.1. *If $n \geq 3$, there is a convex geometry P_n such that $\text{dim}(P_n) = 3$, and $\text{cdim}(P_n) = n + 1$.*

The next result shows that the bound on Dushnik-Miller dimension in the preceding theorem cannot be improved.

Theorem 1.2. *If P is a convex geometry and $\text{dim}(P) \leq 2$, then $\text{cdim}(P) = \text{dim}(P)$.*

Although there are convex geometries with VC-dimension 2 and standard example number 1, we show that these two parameters are essentially the same.

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Theorem 1.3. *If P is a convex geometry, then $\text{vcdim}(P) = \text{se}(P)$ unless $\text{vcdim}(P) = 2$ and $\text{se}(P) = 1$.*

The next result collapses the exceptional case in the preceding theorem.

Theorem 1.4. *If P is a convex geometry and $\text{se}(P) = 1$, then $\text{cdim}(P) \leq 2$.*

The next result separates convex dimension, Dushnik-Miller dimension, Boolean dimension and local dimension from VC-dimension and fractional dimension. In stating this result, we use the abbreviation $[n]$ for $\{1, \dots, n\}$. Also, we denote the base 2 logarithm of n by $\lg n$, while the natural logarithm of n is denoted $\log n$.

Theorem 1.5. *If k and n are integers with $1 \leq k \leq n-2$, there is a family $P(k, n)$ of subsets of $[n]$ such that:*

1. $P(k, n)$ is a convex geometry with ground set $[n]$.
2. If $1 \leq k < k' \leq n-2$, then $P(k, n)$ is a subset of $P(k', n)$.
3. $\text{cdim}(P(k, n)) = \binom{n-1}{k}$.
4. $\text{maxdd}(P(k, n)) = k + 1$.
- 5a. $\text{dim}(P(1, n)) = 1 + \lfloor \lg n \rfloor$,
- 5b. $\text{dim}(P(k, n)) \leq (k+1)2^{k+2} \log n$.
6. $\text{se}(P(k, n)) = k + 1$.
7. For fixed $k \geq 1$, $\text{bdim}(P(k, n)) \rightarrow \infty$ as $n \rightarrow \infty$.
8. For fixed $k \geq 1$, $\text{ldim}(P(k, n)) \rightarrow \infty$ as $n \rightarrow \infty$.
9. $\text{fdim}(P(k, n)) < 2^{k+1}$.

All of the various parameters we are discussing are monotonic on subposets. Therefore, statements 2 and 5a of Theorem 1.5 imply that for fixed $k \geq 1$, $\text{dim}(P(k, n)) \rightarrow \infty$ as $n \rightarrow \infty$.

2. CONVEX GEOMETRIES AND PARTIALLY ORDERED SETS

A *partially ordered set* (we prefer the short form *poset*) is a set P equipped with a binary relation \leq that is reflexive, antisymmetric and transitive. We will assume that readers are familiar with basic concepts for posets including: order diagrams (also called Hasse diagrams); chains and antichains; subposets; maximal and minimal points; isomorphic posets; and linear extensions. We will also assume that readers are familiar with the basics of finite lattices, including zeroes, ones, meets and joins.

In [12], Edelman and Jamison introduce a class of posets they call *convex geometries*. As noted in [12], posets in this class have been studied by various authors, and many combinatorial objects naturally carry the order structure of a convex geometry. Examples include subtrees of a tree [10], convex subsets of a poset [3], convex subgraphs of an acyclic digraph [25], transitively oriented subgraphs of a transitively oriented digraph [5], convex sets of oriented matroids [21], and many more as discussed in [12].

As pointed out in [12], the notation and terminology in the literature for convex geometries is not uniform. As just one example, they have also been called *antimatroids*, e.g., by Korte, Lovász, and Schrader [20]. Accordingly, a unifying framework for convex geometries is developed in [12], and we will follow to a large degree their framework.

Let X be a finite non-empty¹ set. A *convex geometry* with ground set X is a family P of subsets of X satisfying the following three properties:

- (1) $\emptyset, X \in P$. (*Base Property*)
- (2) If A and $B \in P$, then $A \cap B \in P$. (*Intersection Property*)
- (3) If $A \in P$, and $A \neq X$, there exists $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$.
(*Extension Property*)

We refer the reader to [12] and [13] for additional background information on convex geometries, including several equivalent definitions.

There are two special cases of convex geometries which are of particular interest in this paper. As a first example, when L is linear order on a finite non-empty set X , we obtain a convex geometry P from L , with $|P| = 1 + |X|$, by taking P as the set of all initial segments of L , i.e., if $|X| = n$ and $L = \alpha_1 < \dots < \alpha_n$, then P consists of the empty set together with the subsets of X of the form $\{\alpha_1, \dots, \alpha_i\}$, where $i \in [n]$. These simple examples are called *linear geometries*.

As a second example, when X is a finite non-empty set, and P consists of *all* subsets of X , then P is a convex geometry. This special case has been called various names in the literature, with *Boolean algebra* and *subset lattice* being two popular choices. To emphasize the relationship between $|X|$ and $|P|$, we will denote this special case as $P = \mathbf{2}^n$, where $n = |X|$.

As a family of sets, a convex geometry is partially ordered by inclusion, and is therefore a poset. When P is a convex geometry, P is closed under intersections, but in general, P is not closed under unions. Nevertheless, a convex geometry is a lattice. The empty set is the zero, while the ground set X is the one. For sets $A, B \in P$, we have:

$$A \wedge B = A \cap B, \quad \text{and}$$

$$A \vee B = \bigcap \{C \in P : A \cup B \subseteq C\}.$$

Although we will reference results that involve posets in general, our particular focus is on the class of convex geometries.

3. CONCEPTS OF DIMENSION

When P is a poset, we will sometimes use the short form $a \leq_P b$ as a substitute for $a \leq b$ in P . Also, we will write $a \parallel_P b$ when a and b are distinct incomparable elements of P . As our emphasis is combinatorial in the main, and all parameters we study have the same value for two posets that are isomorphic, we will say that $P = Q$ when P and Q are isomorphic posets. In the same spirit, we say P *contains* Q when there is a subposet of P that is isomorphic to Q . We let \mathbf{N} denote the set of all positive integers, and when $n \in \mathbf{N}$, we use $[n]$ to denote the set $\{1, \dots, n\}$ consisting of the least n positive integers.

As it serves to motivate the definitions for the subclass of convex geometries, we elect to begin with four concepts of dimension for the broader class of posets. The first concept of dimension we will investigate is the classic parameter defined by Dushnik and Miller [11]. When $t \geq 1$, a sequence (L_1, \dots, L_t) of linear extensions of a poset P is a *realizer of P* if for all $x, y \in P$, $x \leq_P y$ if and only if $x \leq y$ in L_i for all $i \in [t]$. The *Dushnik-Miller dimension* of P , denoted $\dim(P)$, is the

¹Some authors allow the ground set X to be the empty set, leading to the family $\{\emptyset\}$ to be a convex geometry. Very little of interest can be said about this special case, so we only consider non-empty ground sets.

least positive integer t such that P has a realizer of size t . Following the traditions of the extensive literature on this parameter, it will henceforth simply be called *dimension*.

For each $t \geq 2$, the *standard example* S_t is a height 2 poset with minimal elements a_1, \dots, a_t , maximal elements b_1, \dots, b_t , and order relation $a_i < b_j$ in S_t when $1 \leq i, j \leq t$ and $i \neq j$. As noted in [11], $\dim(S_t) = t$ for all $t \geq 2$. The *standard example number* of a poset P , denoted $\text{se}(P)$, is set to be 1 when P does not contain the standard example S_2 ; otherwise $\text{se}(P)$ is the largest $t \geq 2$ for which P contains the standard example S_t . Evidently, $\dim(P) \geq \text{se}(P)$ for all posets P .

On the one hand, the inequality $\dim(P) \geq \text{se}(P)$ can be far from tight, as among the class of posets which have standard example number 1 (this is the well studied class of interval orders), there are posets that have arbitrarily large dimension. At the other extreme, when P is a poset that is a distributive lattice and $\dim(P) \geq 3$, we have $\dim(P) = \text{se}(P)$.

A class \mathbb{P} of posets is said to be *dim-bounded* if there is a function $f : \mathbf{N} \rightarrow \mathbf{N}$ such that for every $P \in \mathbb{P}$, $\dim(P) \leq f(\text{se}(P))$. We point out the analogous problem in graph theory. Although there are triangle-free graphs with arbitrarily large chromatic number, there are interesting classes of graphs where the chromatic number is bounded in terms of maximum clique size. Such classes are said to be *χ -bounded*.

Blake, Hodor, Micek, Seweryn and Trotter [8] have just resolved a conjecture that is more than 40 years old by showing that the class of posets that have planar cover graphs is dim-bounded. In time, readers will sense how their result prompts many of the questions we address in this paper. For readers who are new to the concept of dimension for posets, compact summaries are given in several recent research papers including [16], [23], [7], and the survey paper [28].

When P is a poset, we let $\text{Inc}(P)$ denote the set of all pairs (a, b) of distinct elements of P with $a \parallel_P b$. Trivially, the following statements are equivalent: (1) $\text{Inc}(P) = \emptyset$; (2) P is a chain; and (3) $\dim(P) = 1$. When $\text{Inc}(P) \neq \emptyset$, and $t \geq 2$, a sequence (L_1, L_2, \dots, L_t) of linear extensions of P is a realizer of P if and only if for every $(a, b) \in \text{Inc}(P)$, there is some $i \in [t]$ such that $a > b$ in L_i .

When S is a non-empty subset of $\text{Inc}(P)$, we say S is *reversible* if there is a linear extension L of P such that $a > b$ in L whenever $(a, b) \in S$. Accordingly, when $\text{Inc}(P) \neq \emptyset$, $\dim(P)$ is the least integer $t \geq 2$ such that $\text{Inc}(P)$ can be covered by t reversible sets.

Now let P be a poset that is not a chain. A pair $(a, b) \in \text{Inc}(P)$ is called a *critical pair* if (1) $x <_P b$ whenever $x <_P a$; and (2) $a <_P y$ whenever $b <_P y$. The set of all critical pairs of P is denoted $\text{crit}(P)$. A sequence (L_1, \dots, L_t) of linear extensions of P is a realizer of P if and only if for every $(a, b) \in \text{crit}(P)$, there is some $i \in [t]$ such that $a > b$ in L_i . Accordingly, $\dim(P)$ is the least integer $t \geq 2$ such that $\text{crit}(P)$ can be covered with t reversible sets.

Again, let P be a poset that is not a chain, and let $k \geq 2$. A sequence $((a_1, b_1), \dots, (a_k, b_k))$ of pairs from $\text{Inc}(P)$ is called a *alternating cycle* in P if $a_i \leq_P b_{i+1}$ for all $i \in [k]$. As suggested by the terminology, this requirement holds cyclically, i.e., we also require $a_k \leq_P b_1$. An alternating cycle $((a_1, b_1), \dots, (a_k, b_k))$ is said to be *strict* if for all $i, j \in [k]$, $a_i \leq_P b_j$ if and only if $j = i + 1$. As is also well known, a non-empty subset $S \subseteq \text{Inc}(P)$ is reversible if and only if there are no strict alternating cycles in P consisting entirely of pairs from S . Also, we note that

when $((a_1, b_1), \dots, (a_k, b_k))$ is a strict alternating cycle, the sets $\{a_i : i \in [k]\}$ and $\{b_i : i \in [k]\}$ are both k -element antichains.

Let P be a poset, and let $\mathcal{R} = (L_1, \dots, L_t)$ be a sequence of linear orders on the ground set of P (these linear orders need not be linear extensions). For a pair (x, y) of distinct elements of P , determine a 0–1 *query string* $q = q(x, y, \mathcal{R})$ of length t by setting $q(i) = 1$ if $x < y$ in L_i ; otherwise, set $q(i) = 0$. The sequence \mathcal{R} is called a *Boolean realizer for P* if there is a set τ of bit strings of length t such that for each pair (x, y) of distinct elements of P , $x <_P y$ if and only if $q(x, y, \mathcal{R}) \in \tau$. As defined by Nešetřil and Pudlák in [24], the *Boolean dimension* of P , denoted $\text{bdim}(P)$, is the least positive integer t for which P has a Boolean realizer of size t . For every poset P , we always have $\text{bdim}(P) \leq \dim(P)$, since a realizer $\mathcal{R} = (L_1, \dots, L_t)$ is a Boolean realizer with $\tau = \{q\}$, where q is a bit string of length t with $q(i) = 1$ for all $i \in [t]$. On the other hand, it is an easy exercise to verify that if $t \geq 2$ and S_t is a standard example, then $\text{bdim}(S_2) = 2$; $\text{bdim}(S_3) = 3$; and $\text{bdim}(S_t) = 4$ for all $t \geq 4$. We refer readers to [22], [15] and [29] for recent results on Boolean dimension.

Let P be a poset. A linear order M is called a *partial linear extension*, abbreviated ple, of P if M is a linear extension of a subposet of P . A sequence (M_1, \dots, M_m) of ple's of P is called a *local realizer of P* if (1) whenever $x \leq_P y$, there is some $i \in [m]$ with $x \leq y$ in M_i ; and (2) whenever $(x, y) \in \text{Inc}(P)$, there is some $j \in [m]$ with $x > y$ in M_j . As defined by Ueckerdt², the *local dimension* of P , denoted $\text{ldim}(P)$, is the least r for which there is a local realizer (M_1, \dots, M_m) of P such that for each $x \in P$, there are at most r different values of $i \in [m]$ with $x \in M_i$. Again, it is clear that $\text{ldim}(P) \leq \dim(P)$ since a realizer is a local realizer. Also, it is an easy exercise to show that if $t \geq 2$ and S_t is the standard example, then $\text{ldim}(S_2) = 2$ and $\text{ldim}(S_t) = 3$ for all $t \geq 3$. We refer readers to [9], [18] and [2] for recent results on local dimension. In particular, [2] contrasts the three notions of dimension we have defined thus far.

It is worth noting that neither of Boolean dimension and local dimension is bounded in terms of the other, as the following results are proved in [29]:

- (1) There is a constant C such that if $\text{ldim}(P) = 3$, then $\text{bdim}(P) \leq C$.
- (2) For every $t \geq 4$, there is a poset P with $\text{ldim}(P) = 4$ and $\text{bdim}(P) \geq t$.
- (3) For every $t \geq 3$, there is a poset P with $\text{bdim}(P) = 3$ and $\text{ldim}(P) \geq t$.

Let P be a poset, and let $\mathcal{E}(P)$ be the set of all linear extensions of P . A *fractional realizer of P* is a function f which assigns non-negative real numbers to the linear extensions in $\mathcal{E}(P)$ such that whenever $(x, y) \in \text{Inc}(P)$, $\sum\{f(L) : L \in \mathcal{E}(P), x > y \text{ in } L\} \geq 1$. As defined by Felsner and Trotter [14], the *fractional dimension* of P , denoted $\text{fdim}(P)$, is the least positive real number t , with $t \geq 1$, such that there is a fractional realizer f with $\sum\{f(L) : L \in \mathcal{E}(P)\} = t$. Now we have $\text{fdim}(P) \leq \dim(P)$ as we can take f to be the 0–1 function assigning value 1 to linear extensions in a realizer of P , while assigning value 0 to all other linear extensions. It is an easy exercise to verify that $\text{fdim}(S_t) = t$ for all $t \geq 2$. On the other hand, it is a nice exercise to show that if $\text{se}(P) = 1$, then $\text{fdim}(P) < 4$. We refer readers to [14] and [4] for additional information and results on fractional dimension.

For the two special cases of convex geometries discussed in the preceding section, we first observe that if P is a linear geometry, then $\dim(P) = \text{bdim}(P) = \text{ldim}(P) =$

²T. Ueckerdt proposed the notion of local dimension at the 2016 Order and Geometry Workshop held in Gultowy, Poland.

$\text{fdim}(P) = \text{se}(P) = 1$. The situation with Boolean algebras is more complex. If $P = \mathbf{2}^n$ is a Boolean algebra, then

- (1) $\dim(P) = \text{fdim}(P) = n$.
- (2) $\text{se}(P) = n$ unless $n = 2$, and in this case $\text{se}(P) = 1$.
- (3) $\text{bdim}(P) = \Omega(n/\log n)$. However, it is not known whether or not $\text{bdim}(P) < n$.
- (4) $\text{ldim}(P) = (1 - o(1))n$.

3.1. VC-Dimension. We now discuss two additional dimension parameters defined for convex geometries but not for posets in general. The first of the two is defined for any non-empty family of sets.

Let X be a finite set, and let F be a non-empty family of subsets of X . The *VC-dimension* of F , denoted $\text{vcdim}(F)$, is defined as follows. Set $\text{vcdim}(F) = 0$ if there is no element $\alpha \in X$ for which there are sets $A, A' \in F$ with $\alpha \in A$ and $\alpha \notin A'$; otherwise, set $\text{vcdim}(F)$ to be the largest $t \geq 1$ for which there is a t -element subset $\{\alpha_1, \dots, \alpha_t\}$ of X such that for every set $S \subset [t]$, there is a set $A = A(S) \in F$ such that $\alpha_i \in A$ if and only if $i \in S$. We note that if P is a linear convex geometry, then $\text{vcdim}(P) = 1$. Also, if P is the Boolean algebra $\mathbf{2}^n$, then $\text{vcdim}(P) = n$.

The next two theorems, both proved in [12], make clear the essential role Boolean algebras play in a discussion of convex geometries.

Theorem 3.1. *Let P be a convex geometry, let X and Y be distinct elements of P such that (1) $Y \neq \emptyset$; and (2) X is the intersection of all sets in P that are covered by Y in P . Then the interval $[X, Y]$ of P is isomorphic to the Boolean algebra $\mathbf{2}^m$, where m is the number of sets that are covered by Y in P .*

Now let Q be a finite lattice. It is natural to ask whether there is a convex geometry P such that $P = Q$. The next theorem, proved in [12], provides an answer.

Theorem 3.2. *Let Q be a finite lattice. Then there is a convex geometry P such P is isomorphic to the poset Q if and only if for every y in Q with $y \neq 0$, if x is the meet of all elements covered by y in Q , then the interval $[x, y]$ in Q is isomorphic to the Boolean algebra $\mathbf{2}^m$, where m is the number of elements of Q that are covered by y .*

In the literature, a finite lattice Q is said to be *meet-distributive* if it satisfies the property in Theorem 3.2. Accordingly, the study of convex geometries can be recast as the study of meet-distributive lattices. One useful property of these lattices is that they are *graded*, i.e., all maximal chains between two comparable elements have the same length.

Motivated by the preceding discussion, we make the following definitions. Let Q be a finite poset, and let $y \in Q$. We denote by $\text{dd}(y, Q)$ the *down degree* of y in Q , i.e., $\text{dd}(y, Q)$ is the number of elements of Q that are covered by y in Q . Of course $\text{dd}(y, Q) = 0$ if and only if y is a minimal element of Q . In turn, we let $\text{maxdd}(Q)$ denote the maximum value of $\text{dd}(y, Q)$ taken over all elements $y \in Q$.

When P is a convex geometry, if $d = \text{maxdd}(P)$, then Theorem 3.1 now implies that P contains the Boolean algebra $\mathbf{2}^d$. Therefore, $\dim(P) \geq \text{vcdim}(P) \geq \text{maxdd}(P)$. The next theorem, proved in [1] shows that this inequality is tight.

Theorem 3.3. *When P is a convex geometry, $\text{vcdim}(P) = \text{maxdd}(P)$.*

For the balance of the paper, we will phrase results in terms of maximum down degree, asking readers to keep in mind that when working with convex geometries, maximum down degree is *exactly* the same as VC-dimension. We make this choice because some of our proofs make direct use of the diagram and the specific value of maximum down degree.

Dually, when Q is a finite poset, and $x \in Q$, we define the *up degree* of x in Q , denoted $\text{ud}(x, Q)$ as the number of elements of Q that cover x . Also, $\text{maxud}(X)$ is the maximum value of $\text{ud}(x, Q)$ taken over all $x \in Q$. As we will soon see, there is no bound on the value of maximum up degree, even when maximum down degree is 2.

3.2. Convex Dimension. The second dimension parameter for convex geometries borrows from the set up for (Dushnik-Miller) dimension. When X is the ground set of a convex geometry P , $|X| = n$, and $L = \alpha_1 < \dots < \alpha_n$ is a linear order on X , we say L is *compatible* when $\{\alpha_1, \dots, \alpha_i\}$ is a set in P for each $i \in [n]$. There is a natural 1-1 correspondence between compatible linear orders on X and maximal chains in P .

Let X be a finite set, let $t \geq 1$, and let (P_1, \dots, P_t) be a sequence of convex geometries each of which has ground set X . Define a family P of subsets of X by:

$$P = \{A_1 \cap \dots \cap A_t : A_i \in P_i \text{ for all } i \in [t]\}.$$

Then P is a convex geometry, and we denote this by writing $P = P_1 \vee \dots \vee P_t$. Furthermore, given a convex geometry P , if we define the sequence (P_1, \dots, P_t) by taking the linear geometries associated with the maximal chains in P , then $P = P_1 \vee \dots \vee P_t$. These observations give rise to the following definitions.

Let P be a convex geometry, and let X be the ground set of P . A sequence (P_1, \dots, P_t) of linear geometries, each with ground set X , is called a *convex realizer* of P if $P = P_1 \vee \dots \vee P_t$. The *convex dimension* of P , denoted $\text{cdim}(P)$, is the least positive integer t for which P has a convex realizer (P_1, \dots, P_t) of size t .

We pause to make an elementary observation. Let $n \geq 2$, let $L_1 = 1 < 2 < \dots < n$, and $L_2 = n < \dots < 2 < 1$. If Q_1 and Q_2 are the linear geometries with ground set $[n]$ determined by L_1 and L_2 , respectively, and $P = Q_1 \vee Q_2$, then $\text{cdim}(P) = \dim(P) = \text{maxdd}(P) = 2$, while $\text{maxud}(P) = n$.

In a finite lattice Q , an element y that satisfies $\text{ud}(y, Q) = 1$ is called a *meet-irreducible element*. This terminology reflects the property that if $y = w \wedge z$, then one of w and z must be y . The following theorem, which provides a *very* useful alternative definition of convex dimension, is proved in [12].

Theorem 3.4. *Let P be a convex geometry. Then $\text{cdim}(P)$ is the width of the subposet of P determined by the meet-irreducible elements of P .*

Next, we present a brief series of elementary results that support more substantive arguments to follow.

Proposition 3.5. *Let Q be a finite lattice. If $(x, y) \in \text{crit}(Q)$, then $\text{dd}(x, Q) = \text{ud}(y, Q) = 1$.*

Proof. We prove that $\text{dd}(x, Q) = 1$. The argument to show that $\text{ud}(y, Q) = 1$ is dual. Suppose to the contrary that $\text{dd}(x, Q) \geq 2$. Then there are distinct elements w and z of Q such that x covers both w and z in Q . Since (x, y) is a critical pair, and $w <_Q x$, we know $w <_P y$. Similarly, we have $z <_P x$, so $z <_P y$. Since Q

is a lattice and $\{x, y\}$ is an antichain, we know that $\{w, z\} <_P qw \vee z <_Q \{x, y\}$. However, these inequalities imply that neither of w and z is covered by x . The contradiction completes the proof. \square

When Q is a finite lattice, and y is a meet-irreducible element of Q , we let $\text{uc}(y, Q)$ denote the unique element of Q that covers y . An element x of Q is called a *join-irreducible* element of Q if $\text{dd}(x, Q) = 1$. In this case, we let $\text{dc}(x, Q)$ denote the unique element of Q covered by x . The next result shows that in a convex geometry P , there is a natural 1–1 correspondence between the set of meet-irreducible elements of P and the set of critical pairs of P .

Proposition 3.6. *Let P be a convex geometry, let B be a meet-irreducible of P , let $Y = \text{uc}(B, P)$, and let $\{\alpha\} = Y - B$. Then the following statements are equivalent:*

- (1) $A = \cap\{U \in P : \alpha \in U\}$.
- (2) $(A, B) \in \text{crit}(P)$.

Proof. Assume that $A = \cap\{U \in P : \alpha \in U\}$. We show that (A, B) is a critical pair. First, since $\alpha \in A \cap Y$, and $\alpha \notin B$, we know $A \not\subseteq B$. If there is an element $\beta \in A - Y$, then $A \cap Y \subsetneq A$, which contradicts the definition of A . We conclude that $A \subset Y$. Furthermore, if $V \subsetneq A$, then $\alpha \notin V$. Since $Y = B \cup \{\alpha\}$, we conclude that $V \subset B$. Now suppose that B is a proper subset of W . Then $Y \subseteq W$. This implies $A \subset W$. These observations imply that (A, B) is a critical pair.

Now suppose that (A, B) is a critical pair. We show that $A = \cap\{U \in P : \alpha \in U\}$. Since $B \subsetneq Y$, we must have $A \subseteq Y$. However, $Y = B \cup \{\alpha\}$, and we conclude that $\alpha \in A$. Now suppose that there is a set $U \in P$ with $\alpha \in U$ and $U \subsetneq A$. Then $U \not\prec_P B$, which contradicts the assumption that (A, B) is a critical pair. We conclude that $A = \cap\{U \in P : \alpha \in U\}$. \square

Proposition 3.7. *Let Q be a finite poset. If $t = \text{se}(Q) \geq 2$, then Q contains a copy of the standard example S_t labeled such that $(a_i, b_i) \in \text{crit}(Q)$ for all $i \in [t]$.*

Proof. Of all copies of the standard example S_t contained in Q , choose one for which the following sum is minimum:

$$\sum_{i \in [d]} |D_Q[a_i]| + |U_Q[b_i]|.$$

Clearly, (a_i, b_i) is a critical pair in Q , for each $i \in [t]$. \square

Proposition 3.8. *Let P be a convex geometry. Then*

$$\text{cdim}(P) \geq \dim(P) \geq \max(\text{maxdd}(P), \text{se}(P)).$$

Proof. The inequalities $\dim(P) \geq \max(\text{maxdd}(P), \text{se}(P))$ are trivial. Let J be the set of meet-irreducibles in P , i.e., J is the set of all sets B for which $\text{ud}(B, P) = 1$. From Theorem 3.4, we know that $\text{cdim}(P)$ is the width of the subposet J . If $\text{cdim}(P) = s$, it follows that there are chains $\mathcal{C}_1, \dots, \mathcal{C}_s$ in P that cover all the sets in J . For each $i \in [s]$, let S_i consist of all critical pairs (A, B) such that $B \in \mathcal{C}_i$. Each set S_i of this form is reversible since it cannot contain a strict alternating cycle. It follows that $\dim(P) \leq s$. \square

4. SEPARATING DIMENSION AND CONVEX DIMENSION

Proposition 3.8 asserts that if P is a convex geometry, then $\text{cdim}(P) \geq \text{dim}(P)$. If P is a linear geometry or the Boolean algebra $\mathbf{2}^n$, then $\text{cdim}(P) = \text{dim}(P)$. More generally, this equality even holds for distributive lattices, see [13]. However, the inequality is not always tight: Figure 1 in [13] illustrates a convex geometry with convex dimension 4 but (Dushnik-Miller) dimension 3.

It is natural to ask whether dimension and convex dimension can be *separated*, i.e., is there a family of convex geometries for which dimension is bounded while convex dimension is not. We now proceed to give an affirmative answer. The remainder of this section constitutes the proof of Theorem 1.1.

As suggested by our notation for Boolean algebras, when n is a positive integer, we let \mathbf{n} denote an n -element chain. Now let n be an integer with $n \geq 3$, and let Q_n be the Cartesian product $\mathbf{n} \times \mathbf{n} \times \mathbf{2}$. Then $|Q_n| = 2n^2$, and $\text{dim}(Q_n) = 3$ for all $n \geq 3$. We show in Figure 1 an order diagram for Q_n (in the particular case when $n = 6$). In this figure, there are $\binom{n-2}{2}$ points that are colored gray, while the remaining points are colored black or white.

We note that $\text{maxdd}(P_n) = 3$, and P_n is a lattice. Furthermore, if y is an element of P_n , y is not the least element of P , and x is the meet of all elements covered by y , then the interval $[x, y]$ of P_n is a Boolean algebra of $\mathbf{2}^m$, where $m = \text{dd}(y, P_n)$. Using Theorem 3.2, we conclude that P_n is a convex geometry.

Using Theorem 3.4, the convex dimension of P_n is $n + 1$. On the other hand, as a subposet of G_n , we know $\text{dim}(P_n) \leq 3$. Since $\text{maxdd}(P_n) = 3$, we know $\text{dim}(P_n) \geq 3$. Therefore, $\text{dim}(P_n) = 3$.

We conclude that we have an infinite family $\{P_n : n \geq 3\}$ of convex geometries with $\text{dim}(P_n) = 3$, and $\text{cdim}(P_n) = n + 1$. Accordingly, we have separated dimension and convex dimension for the class of convex geometries.

5. CONVEX GEOMETRIES WITH DIMENSION 2

The family constructed in the preceding section suggests the following question. Among convex geometries with dimension 2, is convex dimension bounded? In this section, we give an affirmative answer. In fact, we show that when $\text{dim}(P) \leq 2$, $\text{cdim}(P) = \text{dim}(P)$, and the argument constitutes the proof of Theorem 1.2. The statement holds trivially if $\text{dim}(P) = 1$, so we fix a convex geometry P with $\text{dim}(P) = 2$.

Our argument requires an unpublished but by now quite well known result due to K. Baker: A finite lattice has Dushnik-Miller dimension at most 2 if and only if its order diagram is planar. Recall that a poset is said to be planar if its order diagram can be drawn (following all rules for diagrams) without edge crossings in the plane. We show on the left side of Figure 2 a 2-dimensional convex geometry with a planar order diagram.

Using the result of Baker, we take a planar drawing of the order diagram of P . Since P is a lattice, there is a well defined maximal chain in P that constitutes the left boundary in the drawing. Also, there is a well defined maximal chain in P that constitutes the right boundary in the two drawings. These two boundary chains can intersect. Regardless, any element of P that is not on either of these two chains is in the *interior* of the drawing.

Claim 5.1. *The interior faces in the drawing are diamonds.*

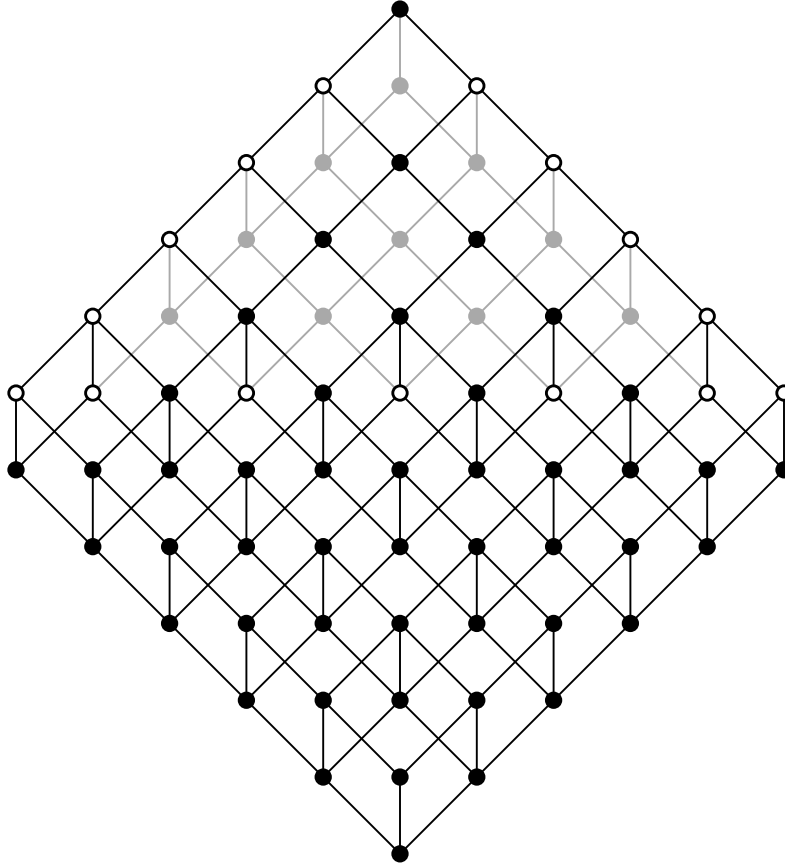


FIGURE 1. For $n \geq 3$, we illustrate a 3-dimensional poset $Q_n = \mathbf{n} \times \mathbf{n} \times \mathbf{2}$ formed by the Cartesian product of three chains. Among the $2n^2$ elements of Q_n , a set of $\binom{n-2}{2}$ points have been coloured grey. Let P_n be the subposet of Q_n obtained by removing the grey points. Then P_n is a convex geometry, and the white points constitute the set of meet-irreducible elements of P_n .

Proof. Since P is a lattice, each face F has a unique maximum element which we denote 1_F . Also, F has a unique minimum element which we denote 0_F . Let A, B be the vertices of F covered by 1_F . By the Boolean property, we know that element $A \cap B$ is covered by A and B , so F is a diamond formed by the elements of $\{1_F, A, B, A \cap B\}$. \square

Claim 5.2. *If A is in the interior of the drawing, then $\text{ud}(A, P) \geq 2$,*

Proof. Referring to the illustration on the right side of Figure 2, suppose to the contrary that A is in the interior of the drawing, and $\text{ud}(A, P) = 1$. Let Y be the unique element of P that covers A . Also, let e be the edge in the drawing having A and Y as its end points. The all points in the plane that belong to e , except possibly Y , are in the interior of the drawing. This implies that e is a boundary

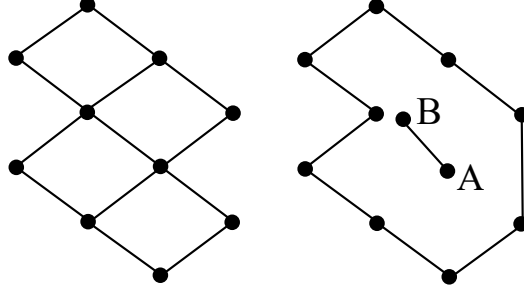


FIGURE 2. On the left, we show a convex geometry with a planar order diagram. Note that maximum down degree is 2, and every interior face is a diamond. On the right, we suggest how an element in the interior would appear if it had up degree 1. We will show that this is impossible.

edge of two interior faces F and F' that have no points in common other than the end points of the edge e .

We note that we cannot have $Y = 1_F$, and $Y = 1_{F'}$, as this would require $\text{dd}(Y, P) \geq 3$. Without loss of generality, we may assume $Y \neq 1_F$. Since F is a diamond, this forces $A = 0_F$. In turn, this forces $\text{ud}(A, P) \geq 2$. The contradiction completes the proof of the claim. \square

To complete the proof of the theorem, we simply observe that the width of the subposet of P consisting of all elements that have up degree 1 in P is 2, since all these elements are on the two boundary chains in the drawing.

6. STANDARD EXAMPLE NUMBER AND MAXIMUM DOWN DEGREE

We have already noted that when P is a convex geometry with ground set X , then $\text{se}(P) \geq \text{maxdd}(P)$, except possibly when $\text{maxdd}(P) = 2$. If $\text{maxdd}(P) = 2$, we can have $\text{se}(P) = 1$, as is the case when P is the Boolean algebra $\mathbf{2}^2$. This raises the question as to whether there is a convex geometry P for which $\text{se}(P) > \text{maxdd}(P)$. We answer this question in the negative. The results of this section constitute the proof of Theorem 1.3.

We argue by contradiction and assume that P is a convex geometry with $\text{se}(P) > \text{maxdd}(P)$. Then $\text{se}(P) \geq 2$. Choose a standard example S_n in P with the elements labeled as $\{A_1, \dots, A_n\} \cup \{B_1, \dots, B_n\}$, so that for all $i, j \in [n]$, $A_i \subseteq B_j$ if and only if $i \neq j$. By Proposition 3.7, we may further assume that for each $i \in [n]$, (A_i, B_i) is a critical pair. From Proposition 3.5, it follows that for each $i \in [n]$, $\text{dd}(A_i, P) = \text{ud}(B_i, P) = 1$.

For each $i \in [n]$, let Y_i be the unique element covering B_i in P . Since $|Y_i - B_i| = 1$, there is a unique element $\alpha_i \in X$ such that $Y_i = B_i \cup \{\alpha_i\}$. Since (A_i, B_i) is a critical pair, and $B_i \subsetneq Y_i$, we know $A_i \subset Y_i$. Since $A_i \parallel_P B_i$, this forces $\alpha_i \in A_i - B_i$.

Now let X_i be the unique set in P that is covered by A_i . Since (A_i, B_i) is a critical pair, we know $X_i \subset B_i$. This now requires $X_i = A_i - \{\alpha_i\}$. i.e., α_i is the only element of A_i that is not in B_i .

Now set $Y_0 = A_1 \vee \dots \vee A_n$. Also, for each $i \in [n]$, let $B'_i = Y_0 \cap B_i$. Then B'_i is an element of P , and $\alpha_i \in A_i - B'_i$. Furthermore, for each $j \in [n]$ with

$j \neq i$, A_j is a subset of B_i and Y_0 . Therefore, $A_j \subset B'_i$. It follows that the sets in $\{A_1, \dots, A_n\} \cup \{B'_1, \dots, B'_n\}$ determine a subposet of P which is isomorphic to S_n . In particular, the sets in $\{B'_1, \dots, B'_n\}$ form an n -element antichain in P .

For each $i \in [n]$, let $Y'_i = B'_i \cup \{\alpha_i\}$. We then have:

$$(1) \quad Y'_i = B'_i \cup \{\alpha_i\} = (Y_0 \cap B_i) \cup \{\alpha_i\} = (Y_0 \cap B_i) \cup Y_0 \cap \{\alpha_i\} \\ = Y_0 \cap \{B_i \cup \{\alpha_i\}\} = Y_0 \cap Y_i.$$

Since Y_0 and Y_i are elements of P , it follows that Y'_i is an element of P . Also, from its definition, it is clear that Y'_i covers B'_i .

Claim 6.1. *If $i, j \in [n]$, then $A_j \subset Y'_i$.*

Proof. If $j \neq i$, then $A_j \subset B'_i \subset Y'_i$. If $j = i$, since $A_i \subset Y_i$ and $A_i \subset Y_0$, it follows from (1) that $A \subset Y'_i$. \square

Claim 6.2. *If $i \in [n]$, then $Y_0 = Y'_i$.*

Proof. The first claim implies that $Y_0 \subseteq Y'_i$ for all $i \in [n]$. If $i \in [n]$, the definition of B'_i implies that $B'_i \subseteq Y_0$. Since Y'_i covers B'_i , we conclude that $Y_0 = Y'_i$. \square

We have now shown the the element Y_0 covers all elements of $\{B'_1, \dots, B'_n\}$. This implies $\max\text{dd}(P) \geq \text{dd}(Y_0, P) \geq n$. In turn, we have $\max\text{dd}(P) \geq n$. With this observation, the proof is complete.

7. CONVEX GEOMETRIES WITH STANDARD EXAMPLE NUMBER 1

We now know that if P is a convex geometry, then $\text{se}(P) = \max\text{dd}(P)$, except possibly when $\text{se}(P) = 1$ and $\max\text{dd}(P) = 2$. In this section, we show that if P is a convex geometry, and $\text{se}(P) = 1$, then $\dim(P) = \text{cdim}(P) \leq 2$. This section constitutes the proof of Theorem 1.4.

We fix a convex geometry P with $\text{se}(P) = 1$. If $\max\text{dd}(P) = 1$, then P is a chain, and the conclusion holds trivially. So we may assume that $\max\text{dd}(P) = 2$.

Let J denote the subposet of P determined by the meet-irreducible elements. Then $\text{cdim}(P) = \text{width}(P)$. If the proposition fails, there is a 3-element antichain $\{B_1, B_2, B_3\}$ in J . For each $i \in [3]$, let Y_i be the unique element of P that covers B_i . Since $\max\text{dd}(P) = 2$, we may assume that $Y_1 \neq Y_2$. Since P does not contain S_1 , either $B_1 <_P Y_2$ or $B_2 <_P Y_1$. Note that since P is a lattice, it cannot be the case that both inequalities hold, so we may assume without loss of generality that $B_1 <_P Y_2$ and $B_2 \parallel_P Y_1$.

Then $Y_1 <_P Y_2$, and there is an element Z_1 covered by Y_2 such that $Y_1 \leq_P Z_1$. The Boolean property then implies that there is an element X covered by Z_1 and B_2 . Furthermore, since P is graded, if $Y_1 <_P Z_1$, then $B_1, Y_1 \parallel_P X$, and $\{B_1, Y_1, B_2, X\}$ determines S_2 . We conclude that $Y_1 = Z_1$.

We note that $\{B_3, Y_3\} \cap \{B_1, Y_1, B_2, Y_2, X\} = \emptyset$, as any common point implies a comparability that does not exist or a down degree larger than two. Furthermore, note that all the Y_i must form a chain, and since Y_2 covers Y_1 and Y_1 has already down degree two, we must have $Y_2 < Y_3$. Let $Y_2 \leq Z_2 < Y_3$ be covered by Y_3 . The Boolean property then implies that there is an element W covered by Z_2 and B_3 . Since Y_2 already has down degree two, and $\{B_1, B_2, B_3\}$ is an antichain, we force $Y_2 < Z_2$. Since P is graded, Y_2 cannot be larger than W . Since $\{B_1, B_2, B_3\}$ is an antichain, B_3 cannot be larger than B_2 . Hence $\{Y_2, B_2, B_3, W\}$ induce a copy of S_2 . The contradiction completes the proof.

8. SEPARATING THREE PARAMETERS FROM THE OTHER FOUR

We have noted that when P is a convex geometry, then $\dim(P) \geq \max\text{dd}(P)$. We note that if P is the Boolean algebra $\mathbf{2}^n$, or a linear geometry, then $\dim(P) = \max\text{dd}(P)$. Note that these parameters also agree on the convex geometries discussed in the preceding two sections. Now it is natural to ask whether there is a class of convex geometries on which maximum down degree is bounded but dimension is not. We now proceed to give an affirmative answer to this question. In fact, we will construct an infinite family of convex geometries for which (1) maximum down degree, fractional dimension, and standard example number are bounded; while (2) convex dimension, dimension, Boolean dimension, and local dimension are unbounded.

When $n \in \mathbf{N}$, we have already adopted the short form $[n]$ for the set $\{1, 2, \dots, n\}$ consisting of the first n integers. Arguments in this section are simplified with the following, admittedly non-standard extension. When m is an integer, and $m \leq 0$, we take $[m] = \emptyset$, i.e., $[m]$ consists of all positive integers α with $\alpha \leq m$.

The remainder of this section constitutes the proof of Theorem 1.5. For the readers convenience, we repeat here the 9 statements of this theorem. Let k and n be integers with $1 \leq k \leq n - 2$. We show there is a family $P(k, n)$ of subsets of $[n]$ satisfying the following properties:

1. $P(k, n)$ is a convex geometry with ground set $[n]$.
2. If $1 \leq k < k' \leq n - 2$, then $P(k, n)$ is a subset of $P(k', n)$.
3. $\text{cdim}(P(k, n)) = \binom{n-1}{k}$.
4. $\max\text{dd}(P(k, n)) = k + 1$.
- 5a. $\dim(P(1, n)) = 1 + \lfloor \lg n \rfloor$,
- 5b. $\dim(P(k, n)) \leq (k + 1)2^{k+2} \log n$.
6. $\text{se}(P(k, n)) = k + 1$.
7. For fixed $k \geq 1$, $\text{bdim}(P(k, n)) \rightarrow \infty$ as $n \rightarrow \infty$.
8. For fixed $k \geq 1$, $\text{ldim}(P(k, n)) \rightarrow \infty$ as $n \rightarrow \infty$.
9. $\text{fdim}(P(k, n)) < 2^{k+1}$.

We begin with the definition of $P(k, n)$. Then we proceed to prove that the nine statements of the theorem are satisfied.

Let k and n be integers with $1 \leq k \leq n - 2$. A set $A \subseteq [n]$ belongs to $P(k, n)$ if and only if the following property is satisfied:

Membership. If $|A| = k + i - 1$, then $[i - 1] \subseteq A$.

With our convention that $[i - 1] = \emptyset$ when $i - 1 \leq 0$, we note that $A \in P(k, n)$ whenever $A \subseteq [n]$ and $|A| \leq k$. On the other hand, $\{1, 2, 3, 6, 11\}$ and $\{1, 2, 6, 10, 11\}$ belong to $P(3, 12)$ while $\{1, 3, 6, 10, 11\}$ does not.

The next result is statement 1.

Proposition 8.1. *The family $P(k, n)$ is a convex geometry with ground set $[n]$.*

Proof. By definition $P(k, n)$ is a family of subsets of $[n]$. We now show that $P(k, n)$ satisfies the Base, Intersection and Extension Properties. First, we observe that \emptyset and $[n]$ satisfy the Membership requirement, so both are in $P(k, n)$. Therefore, $P(k, n)$ satisfies the Base Property.

Let $A, B \in P(k, n)$. We show that $A \cap B \in P(k, n)$. This holds trivially if one of A and B is a subset of the other, so we may assume that neither is contained in the other. The conclusion that $A \cap B \in P(k, n)$ also holds trivially if $|A \cap B| \leq k$,

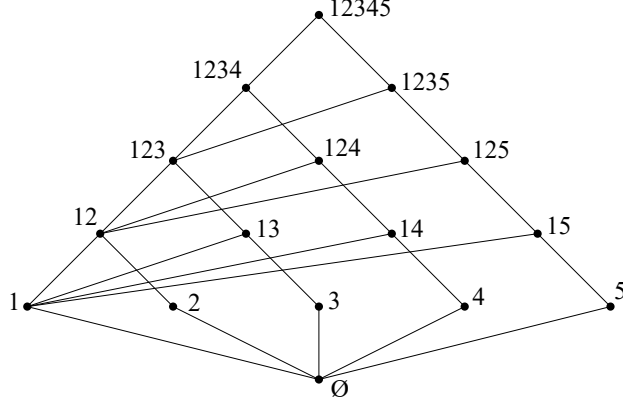


FIGURE 3. We illustrate the convex geometry $P(1,5)$. In the drawing, sets are indicated without braces and commas. Although these facts will follow from the more general arguments given below, readers may enjoy verifying that $\text{cdim}(P(1,5)) = 4$, $\text{dim}(P(1,5)) = \text{bdim}(P(1,5)) = \text{fdim}(P(1,5)) = 3$, and $\text{se}(P(1,5)) = \text{maxdd}(P(1,5)) = 2$.

so we may assume that $|A \cap B| = k + i - 1$ for some i with $2 \leq i \leq n - k + 1$. Then $|A| \geq k + i - 1$ and $|B| \geq k + i - 1$, so $[i - 1] \subset A$ and $[i - 1] \subset B$. Therefore $[i - 1] \subset A \cap B$. Therefore, $A \cap B \in P(k, n)$. With these observations, we have shown that $P(k, n)$ satisfies the Intersection Property.

Finally, let $A \in P(k, n)$ with $A \neq X$. Suppose first that $|A| \leq k$. If $1 \notin A$, then $A \cup \{1\} \in P(k, n)$. If $1 \in A$, then $A \cup \{\alpha\} \in P(k, n)$ for every $\alpha \in [n] - A$.

Now suppose that $|A| = k + i - 1$ for some i with $2 \leq i \leq n - k + 1$. Then $[i - 1] \subset A$. Since $A \neq [n]$, we know $i \leq n - k$. If $i \notin A$, then $A \cup \{i\} \in P(k, n)$. If $i \in A$, then $A \cup \{\alpha\} \in P(k, n)$ for every $\alpha \in [n] - A$. With this observation, we have shown that $P(k, n)$ satisfies the Extension Property. Therefore $P(k, n)$ is a convex geometry with ground set $[n]$. \square

The subfamily $\{P(1, n) : n \geq 3\}$ will play an important role in arguments to follow. We illustrate $P(1, 5)$ in Figure 3 and note that $P(1, 4)$ is the same convex geometry illustrated in Figure 2 in [13].

The next result is statement 2, and it helps to explain our interest in the subfamily $\{P(1, n) : n \geq 3\}$.

Proposition 8.2. *If $1 \leq k < k' \leq n - 2$, then $P(k, n)$ is a subposet of $P(k', n)$.*

Proof. Let $A \in P(k, n)$ and suppose that $|A| = k' + i - 1$. Then $|A| = k + (k' - k) + i - 1$. Since $A \in P(k, n)$, we must have $[k' - k + i - 1] \subset A$. Since $k' - k > 0$, this implies $[i - 1] \subset A$. Therefore, $A \in P(k', n)$. \square

Let $J(k, n)$ denote the family of all sets of the form $[i - 1] \cup B$ where i and B satisfy the following requirements:

- (1) $i \in [n]$ and $B \subset [n]$.
- (2) $i < j$ for every $j \in B$.

- (3) $|B| \leq k$.
 (4) If $|B| < k$, then $B = \{j \in [n] : i + 1 \leq j \leq n\}$.

Proposition 8.3. *If $1 \leq k \leq n - 2$, then $J(k, n)$ is the set of meet-irreducible elements of $P(k, n)$.*

Proof. Clearly, all sets in $J(k, n)$ satisfy the membership requirement and belong to $P(k, n)$. Note further that all sets in $J(k, n)$ have size at least k .

We observe that if $A = [i-1] \cup B$ is in $J(k, n)$ and $|B| < k$. Then $|A| = n-1$, and $[n]$ is the only element of $P(k, n)$ that covers A . On the other hand, if $A = [i-1] \cup B$ is in $J(k, n)$ and $|B| = k$, then the only element of P covering A is $A' = [i] \cup B$. With these observations, we have verified that all elements of $J(k, n)$ are meet-irreducible.

For the converse, suppose A is a meet-irreducible element of $P(k, n)$. Then we must have $|A| \geq k$, for if $|A| < k$, then A is covered by $A \cup \{\alpha\}$ for every $\alpha \in [n-k]$, and recall that $k \leq n-2$ by definition. Also, we note that if $A \in P(k, n)$ and $|A| = n-1$, then A is meet-irreducible. However, given such an element, the membership requirement forces $[n-k-1] \subset A$. If $A = [n] - \{i\}$, then $A = [i-1] \cup B$, where $B = \{j \in [n] : i + 1 \leq j \leq n\}$. This implies $A \in J(k, n)$.

It remains to consider the case where A is a meet-irreducible element of $P(k, n)$ and $k \leq |A| \leq n-2$. Let $|A| = k+i-1$ where $1 \leq i \leq n-k-1$. Then $[i-1] \subseteq A$. If $i \notin A$, then $A \in J(k, n)$. On the other hand, if $i \in A$, then A is covered by $A \cup \{j\}$ for all $j \in [n] - A$. The assumption that $|A| \leq n-2$ then implies that $\text{ud}(A, P(k, n)) \geq 2$. With this observation, the proof is complete. \square

For the balance of this section, when we say that $A = [i-1] \cup B$ is meet-irreducible, we also mean that the requirements for membership in $J(k, n)$ are satisfied by i and B .

The next result is statement 3.

Proposition 8.4. *If $1 \leq k \leq n-2$, then $\text{cdim}(P(k, n)) = \binom{n-1}{k}$.*

Proof. Let $J = J(k, n)$. Then set $w = \binom{n-1}{k}$. The set \mathcal{A} of all meet-irreducible elements of $P(k, n)$ of size k is an antichain in J . Note that a k -element set $A \subset [n]$ belongs to J if and only if $1 \notin A$. It follows that the width of J is at least $|\mathcal{A}| = w$.

To show that $\text{width}(J) \leq w$, we construct an explicit covering of J using w chains. Let B be a k -element subset of $[n]$ with $1 \notin B$, and let $j = \min(B)$. Then $j \geq 2$, and the sets in $\{[i-1] \cup B : 1 \leq i \leq j-1\}$ form a chain of size $j-1$ in $J(k, n)$. Clearly, every element of $J(k, n)$ is contained in a unique chain of this form. \square

The next result includes statement 4.

Proposition 8.5. *Let A be a non-empty set in $P(k, n)$. Then $\text{dd}(A, P(k, n)) = 1$ if and only if A is a singleton. Furthermore, $\text{maxdd}(P(k, n)) = k+1$.*

Proof. Evidently a singleton set $A = \{i\}$ satisfies $\text{dd}(A, P(k, n)) = 1$ since \emptyset is the only set covered by A . Now let A be a set in $P(k, n)$ with $|A| \geq 2$. If $|A| \leq k+1$, then A covers $A_{\{\alpha\}}$ for every $\alpha \in A$. In this case, $\text{dd}(A, P(k, n)) = |A| \geq 2$. Now suppose $|A| = k+i-1$ for some i with $3 \leq i \leq n-k+1$. Then A covers $A - \{\alpha\}$ for every $\alpha \in A$ with $\alpha \geq i-1$. We conclude that $\text{dd}(A, P(k, n)) = k+1$. With this observation, the proof of the first statement of the proposition is complete.

The fact that $\text{maxdd}(P(k, n)) = k+1$ is easily extracted from the proof given in the preceding paragraph. \square

With Proposition 3.6, we noted that there is a 1–1 correspondence between the set $J(k, n)$ of meet-irreducible elements of $P(k, n)$ and the set of critical pairs of $P(k, n)$. Using the fact that all singletons belong to $P(k, n)$, it follows that the critical pairs of $P(k, n)$ consist of all pairs of the form $(\{i\}, [i-1] \cup B)$ where $[i-1] \cup B$ is a meet-irreducible element of $P(k, n)$.

The following proposition is stated for emphasis. It is an immediate consequence of the rule for the 1–1 correspondence.

Proposition 8.6. *Let $A = [i-1] \cup B$ and $A' = [j-1] \cup C$ be meet irreducible elements with $i \leq j$. Also, let (X, Y) and (X', Y') be the critical pairs of $P(k, n)$ associated with A and A' respectively. Then $((X, Y), (X', Y'))$ is a strict alternating cycle of size 2 if and only if $i < j$ and $j \in B$.*

The next proposition is only marginally more complex.

Proposition 8.7. *Let $m \geq 2$, and let (A_1, \dots, A_m) be a sequence of meet-irreducible elements. For each $j \in [m]$, let (X_j, Y_j) be the critical pair associated with A_j . If the sequence $((X_1, Y_1), \dots, (X_m, Y_m))$ is a strict alternating cycle, then $m = 2$.*

Proof. For each $j \in [m]$, let $A_j = [i_j - 1] \cup B_j$. Then $(X_j, Y_j) = (\{i_j\}, A_j)$. Since the alternating cycle is strict, the elements of $\{i_1, \dots, i_m\}$ are distinct. With a relabeling if necessary, we may assume that $i_1 < i_j$ for all $j = 2, \dots, m$.

It follows that for each $j = 2, \dots, m$, $X_1 = \{i_1\} \subset [i_j - 1] \subset A_j = Y_j$. Again, since the alternating cycle is strict, this now requires $m = 2$. \square

If $1 \leq k \leq n - 2$ and t is a positive integer, a sequence (Y_1, \dots, Y_n) of subsets of $[t]$ is said to be (k, n) -*distinguishing* if for every meet-irreducible element $A = [i-1] \cup B \in J(k, n)$, there is an element $\alpha \in Y_i$ such that $\alpha \notin Y_j$ whenever $j \in B$.

Proposition 8.8. *If $1 \leq k \leq n - 2$, $\dim(P(k, n))$ is the least positive integer t for which there is a (k, n) -distinguishing sequence (Y_1, \dots, Y_n) of subsets of $[t]$.*

Proof. Suppose first that $\dim(P(k, n)) = t$. We show that there is a sequence (Y_1, \dots, Y_n) of subsets of $[t]$ that is (k, n) -distinguishing. Let S_1, \dots, S_t be reversible sets covering all critical pairs of $P(k, n)$. For each $i \in [n]$, let Y_i consist of all $\alpha \in [t]$ such that there is a critical pair $(X, Y) = (\{i\}, [i-1] \cup B)$ assigned to S_α . We claim that (Y_1, \dots, Y_n) is (k, n) -distinguishing. To see this, let $[i-1] \cup B$ be any meet-irreducible element of $P(k, n)$. Then there is some $\alpha \in [t]$ such that the critical pair $(X, Y) = (\{i\}, [i-1] \cup B)$ is assigned to S_α . We claim that $\alpha \notin Y_j$ for all j with $j \in B$. If this fails, there is a meet-irreducible element $[j-1] \cup C$ such that the critical pair $(X', Y') = (\{j\}, [j-1] \cup C)$ is also assigned to S_α . However, $i < j$ and $j \in B$ imply that $((X, Y), (X', Y'))$ is a strict alternating cycle, contradicting the fact that S_α is reversible. We conclude that the sequence (Y_1, \dots, Y_n) is (k, n) -distinguishing as claimed.

For the converse, suppose that t is a positive integer, and there is a sequence (Y_1, \dots, Y_n) of subsets of $[t]$ that is (k, n) -distinguishing. We show that $\dim(P(k, n)) \leq t$. To accomplish this, for each $\alpha \in [t]$, we let S_α consist of all critical pairs $(X, Y) = (\{i\}, [i-1] \cup B)$ such that $\alpha \in Y_i$ and $\alpha \notin Y_j$ whenever $j \in B$. Using Propositions 8.6 and 8.7, for each $\alpha \in [t]$, the set S_α is reversible, so $\dim(P(k, n)) \leq t$. \square

For each convex geometry in the family $\{P(1, n) : n \geq 3\}$, we now determine the value of the dimension *exactly*. The following result is statement 5a.

Proposition 8.9. *If $n \geq 3$, then $\dim(P(1, n)) = 1 + \lfloor \lg n \rfloor$.*

Proof. Let $t = \dim(P(1, n))$ and $s = 1 + \lfloor \lg n \rfloor$. Then there is a $(1, n)$ -distinguishing sequence (Y_1, \dots, Y_n) of subsets of $[t]$. When $1 \leq i < j \leq n$, the meet-irreducible set $[i-1] \cup \{j\}$ requires $Y_i - Y_j \neq \emptyset$. In particular, this requires $Y_i \neq Y_j$ and $Y_i \neq \emptyset$. The meet-irreducible set $[n-1]$ requires $Y_n \neq \emptyset$. We have now shown that the sets in the sequence (Y_1, \dots, Y_n) are distinct and non-empty. This requires $t \geq s$.

For the converse, consider the family of all subsets of $[s]$ partially ordered by inclusion. Set $m = 2^s$, and consider an arbitrary linear extension L of the Boolean algebra $\mathbf{2}^s$. Then let (Y_1, \dots, Y_n) be the dual of L . Note that $Y_1 = [s]$ and $Y_m = \emptyset$. Since $n < 2^s$, the last set Y_n in the subsequence (Y_1, \dots, Y_n) is non-empty. It follows that this sequence is $(1, n)$ -distinguishing. Therefore, $t \leq s$. \square

The following result is statement 5b. The proof is an elementary application of the probabilistic method.

Proposition 8.10. *For $2 \leq k \leq n - 2$, $\dim(P(k, n)) \leq (k + 1)2^{k+2} \log n$.*

Proof. Let $t = \lfloor (k + 1)2^{k+2} \log n \rfloor$. Then consider a sequence $\sigma = (Y_1, Y_2, \dots, Y_n)$ of random subsets of $[n]$, i.e. for each pair (i, α) with $i \in [n]$ and $\alpha \in [t]$, we assign α to the set Y_i with probability $1/2$. Assignments made for distinct pairs are independent. The probability that σ fails to be (k, n) -distinguishing is at most:

$$(2) \quad n^{k+1} \left(1 - \frac{1}{2^{t+1}}\right)^t$$

Simple calculations show that for the specified value of t , the expression given in (2) is less than 1. It follows that a (k, n) -distinguishing sequence $\sigma = (Y_1, \dots, Y_n)$ exists. This implies that $\dim(P(k, n)) \leq t$. \square

Statement 6 follows from Theorem 1.3. However, there is a direct proof which is more revealing of the structure of these geometries.

Proposition 8.11. *If $1 \leq k \leq n - 2$, $\text{se}(P(k, n)) = k + 1$.*

Proof. For each $i \in [k+1]$, let (X_i, Y_i) be the critical pair $(\{i\}, [i-1] \cup \{i+1, \dots, i+k\})$. Then the elements of $\{X_1, \dots, X_{k+1}\}$ and $\{Y_1, \dots, Y_{k+1}\}$ form the standard example S_{k+1} . This shows that $\text{se}(P(k, n)) \geq k + 1$.

Now suppose that $P(k, n)$ contains a standard example of size $k + 2$. Then there is a sequence $((X_1, Y_1), \dots, (X_{k+2}, Y_{k+2}))$ of critical pairs of $P(k, n)$ such that whenever $1 \leq i < j \leq k + 2$, the pairs (X_i, Y_i) and (X_j, Y_j) form a copy of the standard example S_2 . For each $j \in [m]$, let $X_j = \{i_j\}$ and $Y_j = [i_j - 1] \cup B_j$. Then the elements of $\{i_1, \dots, i_{k+2}\}$ are all distinct. After a relabelling, we may assume that $i_1 < i_2 < \dots < i_{k+2}$. Then $i_j \in B_1$ for $j = 2, \dots, k + 2$. This is impossible since $|B_1| \leq k$. The contradiction completes the proof. \square

The next two results are statements 7 and 8. In the proofs, when we write $\{i_1, i_2, \dots, i_m\}$ is an m -element subset of $[n]$, we imply $i_1 < i_2 < \dots < i_m$. Readers will note that our proofs use Ramsey theory and borrow from ideas in [2].

Proposition 8.12. *For fixed $k \geq 1$, $\text{bdim}(P(k, n)) \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. In view of Proposition 8.2, it suffices to prove the result when $k = 1$. We assume that there is an integer $t \in \mathbf{N}$ such that $\text{bdim}(P(1, n)) \leq t$ and argue to a contradiction provided n is sufficiently large in terms of t . Let $\mathcal{R} = (L_1, \dots, L_t)$ be a Boolean realizer for $P(1, n)$.

Now let $\{i_1, i_2, i_3\}$ be a 3-element subset of $[n]$. Then $A = [i_1 - 1] \cup \{i_2\}$ and $B = [i_2 - 1] \cup \{i_3\}$ are meet-irreducible elements of $P(k, n)$. Also, $A \parallel B$ in $P(1, n)$. The query string $q(A, B, \mathcal{R})$ is a bit string of length t , so we have defined a coloring of the 3-element subsets of $[n]$ using 2^t colors. It follows that if n is sufficiently large, then there is a 4-element subset $H = \{i_1, i_2, i_3, i_4\}$ of $[n]$ such that each of the 3-element subsets of H is assigned the same color. Let $A = [i_1 - 1] \cup \{i_2\}$, $B = [i_2 - 1] \cup \{i_3\}$ and $C = [i_3 - 1] \cup \{i_4\}$. It follows that $q(A, B, \mathcal{R}) = q(B, C, \mathcal{R})$. This implies that $q(A, C, \mathcal{R}) = q(A, B, \mathcal{R})$. This is a contradiction since $A < C$ in $P(1, n)$. \square

Proposition 8.13. *For fixed $k \geq 1$, $\text{ldim}(P(k, n)) \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. In view of Proposition 8.2, it suffices to prove the result when $k = 1$. We assume that there is an integer $t \in \mathbf{N}$ such that $\text{ldim}(P(1, n)) \leq t$ and argue to a contradiction *provided* n is sufficiently large in terms of t . Let $\mathcal{R} = (L_1, \dots, L_m)$ be a local realizer of $P(1, n)$ such that each element of $P(1, n)$ appears in at most t different extensions in this list.

For each 4-element subset $\{i_1, i_2, i_3, i_4\}$ of $[n]$, we consider the meet-irreducible element $A = [i_1 - 1] \cup \{i_3\}$ and the meet-irreducible element $B = [i_2 - 1] \cup \{i_4\}$. Then $A \parallel B$ in $P(1, n)$. Let s be the least integer in $[m]$ such that $A > B$ in L_s . Then there is a uniquely determined pair (r, r') of integers in $[t]$ such that occurrence r of A is in L_s and occurrence r' of B is in L_s .

Now we are coloring the 4-element subsets of $[n]$ with t^2 colors. It follows that if n is sufficiently large, then there is a 7-element subset $H = \{i_1, \dots, i_7\}$ such that each 4-element subset of H is assigned the same color, say (r, r') , where $r, r' \in [t]$.

Let $A = [i_1 - 1] \cup \{i_4\}$, $B = [i_2 - 1] \cup \{i_5\}$, $C = [i_3 - 1] \cup \{i_6\}$, and $D = [i_5 - 1] \cup \{i_7\}$. Let s be the least integer in $[m]$ such that $A > B$ in L_s . Using the set $\{i_1, i_2, i_4, i_5\}$, we know that occurrence r of A is in L_s , and occurrence r' of B is in L_s . Using the set $\{i_1, i_3, i_4, i_6\}$, we know that occurrence r' of C is also in L_s with $A > C$ in L_s . Using the set $\{i_2, i_3, i_5, i_6\}$, we know that occurrence r of B is in L_s with $B > C$ in L_s . This forces $r = r'$.

Using the set $\{i_3, i_5, i_6, i_7\}$, we conclude that occurrence r of C is over occurrence r of D in L_s . This forces $A > D$ in L_s , which is a contradiction since $A < D$ in $P(1, n)$. The contradiction completes the proof. \square

The final result in this section is statement 9.

Proposition 8.14. *If $1 \leq k \leq n - 2$, then $\text{fdim}(P(k, n)) < 2^{k+1}$.*

Proof. We show that $\text{fdim}(P) < 2^{k+1}$ by constructing an appropriate fractional realizer f .

Let $m = 2^n$, and let $\sigma = (Z_1, \dots, Z_{m-1})$ be a listing of the non-empty subsets of $[n]$. The order of the sets in the list is arbitrary. Then for each $\alpha \in [m - 1]$, there is a linear extension L_α of P that reverses all critical pairs of the form $(\{i\}, [i - 1] \cup B)$, where $i \in Z_\alpha$ and $j \notin Z_\alpha$ whenever $i < j$ and $j \in B$.

Then we set $f(L_\alpha) = 2^{k+1}/m$ for each $\alpha \in [m - 1]$. Also, we set $f(L) = 0$ for all other linear extensions of P .

We now show that f is a fractional realizer of P . We need only show that if $C, D \in P(k, n)$ and $C \parallel_P D$, then the following inequality holds:

$$(3) \quad s(C, D) = \sum \{f(L_\alpha) : \alpha \in [m - 1], C > D \text{ in } L_\alpha\} \geq 1.$$

Choose a critical pair (X, Y) such that $X \leq C$ in $P(k, n)$ and $D \leq Y$ in $P(k, n)$. If $\alpha \in [m - 1]$ and $X > Y$ in L_α , then $C > D$ in L_α . It follows that we need only show that inequality (3) holds for every critical pair.

Let $(X, Y) = (\{i\}, [i - 1] \cup B)$ be a critical pair. Then number of sets in the sequence σ that contain i but do not contain j when $i < j$ and $j \in B$ is exactly $2^{n-|B|-1}$, which is at least 2^{n-k-1} . It follows that

$$s(X, Y) \geq 2^{n-k-1} \cdot \frac{2^{k+1}}{m} = 1$$

On the other hand, we note that

$$\sum_{i=1}^{m-1} f(L_i) = 2^{k+1} \frac{m-1}{m} < 2^{k+1}.$$

We conclude that $\text{fdim}(P(k, n)) < 2^{k+1}$. □

9. OPEN PROBLEMS

For posets in general, the standard examples have bounded Boolean dimension, bounded local dimension, and unbounded dimension. As noted previously, the constructions given in [29] show that neither of Boolean dimension and local dimension is bounded in terms of the other. However, for convex geometries, we have been unable to separate dimension from either of Boolean dimension and local dimension. Also, we have been unable to separate Boolean dimension and local dimension in either direction. We note that for the special class of distributive lattices, large dimension forces large Boolean dimension and large local dimension, so separating these parameters is not possible. On the other hand, many other special classes of convex geometries have been identified in the literature, and we wonder whether some of them exhibit properties that enable these questions to be answered.

For the family $\{P(k, n); 1 \leq k \leq n - 2\}$ of convex geometries discussed in the preceding section, we determined the value of $\text{dim}(P(1, n))$ exactly. When $k \geq 2$, we feel it is unlikely that we can obtain an exact formula for $\text{dim}(P(k, n))$. However, a lower bound of the form $c2^k \log n$ where c is a positive constant should hold.

We suspect that the upper bound $\text{fdim}(P(k, n)) < 2^{k+1}$ is asymptotically tight, i.e., for every $\epsilon > 0$, $\text{fdim}(P(k, n)) > 2^{k+1} - \epsilon$ when n is sufficiently large.

Fractional dimension is the linear programming relaxation of dimension. Although not studied in the main body of this paper, there is a natural linear programming relaxation of local dimension, called *fractional local realizer*. We refer the reader to [26] for the precise definition. We wonder whether the fractional local dimension of the convex geometry $P(k, n)$ is always strictly less than its fractional dimension.

Let us finally mention that there are natural classes of posets beyond convex geometries in which VC-dimension and convex dimension may be defined, and we do not know if the different concepts of dimension that are tied in convex geometries may be separated. Most generally one may take an isometric subgraph of the hypercube, root it at a vertex v and order vertices $u \leq w$ if there is a shortest path from v through u to w . Such posets arise from tope graphs of complexes of oriented matroids (see [19, Chapter 7.6] for several related questions) and have been investigated for oriented matroids by Björner, Edelman, and Ziegler [6].

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