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ON THE COMPLEXITY OF POSETS

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The purpose of this paper is to discuss several invariants each of which provides a measure of the intuitive notion of complexity for a finite partially ordered set. For a poset **X**, the invariants discussed include cardinality, width, length, breadth, dimension, weak dimension, interval dimension, and semiorder dimension, denoted respectively |**X**|, W(**X**), L(**X**), B(**X**), dim(**X**), Wdim(**X**), Idim(**X**), and Sdim(**X**). Among these invariants the following inequalities hold. B(**X**) ≤ Idim(**X**) ≤ Sdim(**X**) ≤ Wdim(**X**) ≤ dim(**X**) ≤ W(**X**). We prove that every poset **X** with three or more points contains a pair x, y with Idim(**X**) ≤ 1 + Idim(**X** − {x, y}). If M denotes the set of maximal elements and A an arbitrary antichain of **X**, we show that Idim(**X**) ≤ W(**X** − M) and Idim(**X**) ≤ 2W(**X** − A) + 1. We also show that there exist functions $f(n, t)$ and $g(t)$ such that L(**X**) ≤ n and Idim(**X**) ≤ t imply dim(**X**) ≤ $f(n, t)$ and Sdim(**X**) ≤ t implies dim(**X**) ≤ $g(t)$.

1. Preliminaries

In this paper we consider a *poset* as a pair (X, P) where X is a finite set and P is an irreflexive, transitive (and thus asymmetric) relation on X. P is called a *strict partial order* on X. The notations $(x, y) \in P$, xPy, $x < Y$ in P, and $y > x$ in P are used interchangeably. The notation $x \leq y$ ($x \geq y$) in P means either $x < y$ ($x > y$) in P or $x = y$. For distinct points $x, y \in X$, if neither (x, y) nor (y, x) is in P, we say x and y are *incomparable* and write xIy. We also define $\mathcal{I}_P = \{(x, y) : xIy\}$. If $\mathcal{I}_P = \emptyset$, then we call P a *linear order*. If P and Q are partial orders on X and $P \subsetneq Q$, Q is called an *extension* of P. By a theorem of Szpilrajn [13], if $\mathcal{E}$ denotes the collection of all linear extensions of P, then $\bigcap \mathcal{E} = P$. For convenience, we will frequently use the single symbol **X** to denote the poset (X, P). If $Y \subsetneq X$, the poset $(Y, P \cap (Y \times Y))$ is called a *subposet* of (X, P). When we use a single symbol, the statement **Y** is a subposet of **X**, denoted **Y** ⊆ **X**, means that the partial order on Y is the intersection of the partial order on X with $Y \times Y$. A subposet which is a linear order is called a *chain*. We denote the n-element chain $0 < 1 < 2 < \ldots < n - 1$ by **n**. A subposet for which every distinct pair of points is an incomparable pair is called an *antichain*. The *length* L(**X**) and the *width* W(**X**) are the cardinality of a maximum chain and a maximum antichain in **X** respectively. We use $|\mathbf{X}|$ to denote the cardinality of **X**.

If (X, P) and (Y, Q) are posets, then the *free sum* of (X, P) and (Y, Q), denoted $(X, P)+(Y, Q)$ or $\mathbf{X}+\mathbf{Y}$, is the poset $(X \dot\cup Y, P \dot\cup Q)$ where $\dot\cup$ denotes the disjoint union of sets.

If (X, P) and (Y, Q) are posets, then the *cartesian product* of (X, P) and (Y, Q), denoted $(X, P) \times (Y, Q)$ or $\mathbf{X} \times \mathbf{Y}$ is the poset $(X \times Y, S)$ where $(x, y) \leq (z, w)$ in S when $x \leq z$ in P and $y \leq w$ in Q. We will use **R** to denote the set of real numbers with the usual ordering and $\mathbf{R}^n$ to denote the cartesian product of n copies of **R**.

If (X, P) and (Y, Q) are posets, then the *join* of (X, P) and (Y, Q), denoted $(X, P) \oplus (Y, Q)$ or $\mathbf{X} \oplus \mathbf{Y}$ is the poset $(X \dot\cup Y, S)$ where $S = P \cup Q \cup (X \times Y)$.

In later sections of this paper, we will frequently be faced with the problem of constructing extensions of partial orders. Consequently we will find it convenient to develop a criterion by which we can determine whether there exists an extension Q of a partial order P so that Q contains a given set $S \subseteq \mathcal{I}_P$. Clearly this problem reduces to determining whether the transitive closure of the relation $P \cup S$, denoted $\overline{P \cup S}$, is a partial order: we note that $\overline{P \cup S}$ is a partial order if and only if it is irreflexive.

If (X, P) is a poset and $S \subseteq \mathcal{I}_P$, then a subset $\{(a_i, b_i): 1 \leq i \leq m\} \subseteq S$ for which $b_i \leq a_{i+1}$ in P (the subscripts are interpreted cyclically, i.e. $a_{m+1} = a_1$) is called a *weak TM-cycle*. The integer m is called the length of the weak TM-cycle. If S contains a weak TM-cycle, then $\overline{P \cup S}$ is not a partial order since it fails to be irreflexive. The converse of this statement is also true; we refer the reader to [19] for a proof of this elementary result.

Lemma 1.1. *Let (X, P) be a poset and let $S \subseteq \mathcal{I}_P$. Then $\overline{P \cup S}$ is a partial order if and only if S does not contain a weak TM-cycle.*

A subset $\{(a_i, b_i): 1 \leq i \leq m\} \subseteq \mathcal{I}_P$ is called a *strong TM-cycle* when $b_i \leq a_j$ in P iff $j = i + 1$ for all i with $1 \leq i \leq m$. (As before we intend for the subscripts to be interpreted cyclically.)

Lemma 1.2. *Let (X, P) be a poset and let $S \subseteq \mathcal{I}_P$. Then $\overline{P \cup S}$ is a partial order iff S does not contain a strong TM-cycle.*

Proof. It suffices to show that a subset $S \subseteq \mathcal{I}_P$ which contains a weak TM-cycle also contains a strong TM-cycle. To accomplish this we choose a weak TM-cycle $\{(a_i, b_i): 1 \leq i \leq m\} \subseteq S$ so that the length m of the cycle is as small as possible. Now $a_1 \geq b_m$ in P and $a_1 I b_1$ in P. Suppose however that there exists an integer with $2 \leq n < m$ so that $a_1 \geq b_n$ in P. It would then follow that $\{(a_i, b_i): 1 \leq i \leq n\}$ is a weak TM-cycle of length n. The contradiction shows that $a_1 \not\geq b_i$ for all i with $1 \leq i < m$. We may use the natural symmetry to conclude that $b_i \leq a_j$ in P iff $j = i + 1$ for all i with $1 \leq i \leq m$.

Let C be a chain in a poset (X, P), $S_1 = \{(c, x): c \in C$ and cIx in $P\}$, and $S_2 = \{(x, c): c \in C$ and xIc in $P\}$. It follows immediately from Lemma 1.2 that $\overline{P \cup S_1}$ and $\overline{P \cup S_2}$ are partial orders. A linear extension L_1 of $\overline{P \cup S_1}$, is called an ***upper extension of*** C and a linear extension L_2 of $\overline{P \cup S_2}$ is called a ***lower extension of*** C.

If A and B are disjoint subsets of X, then we define an *injection of B over A* as a linear extension of $\overline{P \cup S}$ where $S = \mathcal{I}_P \cap (A \times B)$. In this terminology, an upper extension of a chain C is an injection of $X - C$ over C.

Lemma 1.3. *Let A and B be disjoint subsets of a poset (X, P). Then there exists an injection of B over A if and only if $\mathcal{I}_P \cap (A \times B)$ does not contain a strong TM-cycle of length 2.*

Proof. Let $\{(a_i, b_i): 1 \le i \le m\}$ be a strong TM-cycle of length m contained in $\mathcal{I}_P \cap (A \times B)$. If $m > 2$, then $\{(a_1, b_1), (a_2, b_m)\}$ is a strong TM-cycle of length 2 contained in $\mathcal{I}_P \cap (A \times B)$.

The reader should note if $\mathcal{I}_P \cap (A \times B)$ contains a strong TM-cycle $\{(a_i, b_i): 1 \le i \le 2\}$ of length 2, then the points a_1, a_2, b_1 and b_2 are distinct. They form a subposet of (X, P) isomorphic to $\mathbf{2} + \mathbf{2}$.

2. Mathematical formulation of complexity for posets

Dushnik and Miller [7] defined the dimension of a poset (X, P), denoted $\dim(X, P)$ or $\dim(\mathbf{X})$, as the minimum number of linear extensions of P whose intersection is P. The dimension of a poset can be interpreted as a measure of the complexity of the poset in the following sense. Suppose each of a finite number of observers expresses his individual opinion on the relative merits of a finite set of options by ranking the options in a linear order. A partial ordering on the options is obtained by ranking option x higher than option y when all observers have agreed that x is preferred to y. Conversely, the dimension of a partial order measures the minimum number of observers necessary to produce the given partial order as a statement of those preferences on which the observers agree unanimously.

We observe that certain elementary invariants such as cardinality, width, and length can also be interpreted as measures of complexity. However it is clear that this interpretation is limited in that these invariants may prescribe an inordinately high degree of complexity to such intuitively simple partial orders as chains and antichains.

In this paper, we will discuss other measures of complexity for finite partial orders. Each of these measures will be a variant of the concept of dimension. We will note the mathematical and conceptual advantages (and disadvantages) of each.

A poset (X, P) is called a *weak order* if and only if there exists a function

$f: \mathbf{X} \to \mathbf{R}$ so that $x < y$ in P iff $f(x) < f(y)$ in $\mathbf{R}$. It is elementary to prove the following characterization theorem for weak orders.

Theorem 2.1. *A poset is a weak order if and only if it does not contain a subposet isomorphic to* $\mathbf{2} + \mathbf{1}$.

It is natural then to define the *weak dimension* of a poset (X, P), denoted $\mathrm{Wdim}(X, P)$ or $\mathrm{Wdim}(\mathbf{X})$ as the smallest positive integer k for which there exists a function $f: \mathbf{X} \to \mathbf{R}^k$ so that $x < y$ in $\mathbf{X}$ iff $f(x)(i) < f(y)(i)$ in $\mathbf{R}$ for all i with $1 \leq i \leq k$. The function f is called a *point coordinatization of length k* of the poset (X, P).

Ore [10] gave the following alternate definition of the dimension of a poset. $\dim(X, P)$ is the smallest positive integer k for which (X, P) is isomorphic to a subposet of the cartesian product of k chains. For finite posets, it is then easy to see that $\dim(X, P)$ is the smallest integer k for which (X, P) is isomorphic to a subposet of $\mathbf{R}^k$, i.e. $\dim(X, P)$ is the smallest positive integer k for which there exists a function $f: X \to \mathbf{R}^k$ so that $x \leq y$ in P iff $f(x)(i) \leq f(y)(i)$ for all i with $1 \leq i \leq k$. Thus we see that $\mathrm{Wdim}(\mathbf{X}) \leq \dim(\mathbf{X})$ for every poset $\mathbf{X}$. Furthermore it is trivial to verify that $\mathrm{Wdim}(\mathbf{X}) = \dim(\mathbf{X})$ unless $\mathbf{X}$ is a weak order but not a chain, and in this case $\mathrm{Wdim}(\mathbf{X}) = 1$ while $\dim(\mathbf{X}) = 2$.

A collection $\mathscr{C}$ of closed intervals (points are also considered closed intervals) of $\mathbf{R}$ has a natural ordering $<$ induced on it by $A < B$ iff $x \in A$, $y \in B$ implies $x < y$ in $\mathbf{R}$. Any poset which is isomorphic to a poset of the form $(\mathscr{C}, <)$ is called an *interval order*. We state a well known theorem of Fishburn [8] which gives a characterization of interval orders.

Theorem 2.2. *A poset* $\mathbf{X}$ *is an interval order if and only if it does not contain a subposet isomorphic to* $\mathbf{2} + \mathbf{2}$.

A poset (X, P) is called a *semiorder*[1] when there exists a real number d and a function $f: \mathbf{X} \to \mathbf{R}$ so that $x < y$ in $\mathbf{X}$, iff $f(x) + d < f(y)$ in $\mathbf{R}$. It is easy to see that a poset is a semiorder iff it is isomorphic to an interval order in which all intervals have unit length. The following well known characterization of semiorders is due to Scott and Suppes [12].

Theorem 2.3. *An interval order* $\mathbf{X}$ *is a semiorder if and only if it does not contain a subposet isomorphic to* $\mathbf{3} + \mathbf{1}$.

We note that the join of two interval orders (semiorders) is another interval order (semiorder).

[1] Such posets are discussed in a different setting by Dean and Keller [5] who showed that the number of semiorders on n points is $\binom{2n}{n} / (n+1)$. They called semiorders natural partial orders.

Using the notions of interval orders and semiorders, we can now define two new invariants for a finite poset. First we define the *interval dimension* of $\mathbf{X}$, denoted $\operatorname{Idim}(\mathbf{X})$, as the smallest positive integer k for which there exists a function F which assigns to each $x \in \mathbf{X}$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(k)$ of closed intervals of $\mathbf{R}$ such that $x < y$ in $\mathbf{X}$ iff $F(x)(i) < F(y)(i)$ for all $i \leq k$. F is called an *interval coordinatization* of $\mathbf{X}$ of length k. To define the *semiorder dimension* of $\mathbf{X}$, denoted $\operatorname{Sdim}(\mathbf{X})$, we further require that $F(x)(i)$ have length 1 for all $x \in \mathbf{X}$ and all $i \leq k$. As an immediate consequence of these definitions, we have $\operatorname{Idim}(\mathbf{X}) \leq \operatorname{Sdim}(\mathbf{X})$ for all posets $\mathbf{X}$. If $\mathbf{X}$ is an antichain, then $\operatorname{Idim}(\mathbf{X}) = \operatorname{Sdim}(\mathbf{X}) = \operatorname{Wdim}(\mathbf{X}) = 1$. If $\mathbf{X}$ is not an antichain, choose a point coordinatization f of $\mathbf{X}$ of length k and let $m = \min\{f(x)(i) - f(y)(i) : i \leq k, x > y\}$. Then $\mathbf{X}$ has a unit interval coordinatization defined by $F(x)(i) = [2f(x)(i)/m, 1 + 2f(x)(i)/m]$ and thus $\operatorname{Sdim}(\mathbf{X}) \leq \operatorname{Wdim}(\mathbf{X})$ for all $\mathbf{X}$.

We also note that the authors and Rabinovitch have recently proven [3] that for each $n \geq 1$, there exists a poset $\mathbf{X}$ with $\operatorname{Idim}(\mathbf{X}) = 1$ and $\dim(\mathbf{X}) = n$. This proof is easily modified to obtain an analogous result for semiorder dimension.

We state the following elementary result which is easily proved by induction. We refer to this result as the interpolation lemma.

Lemma 2.4. *Let (X, P) be a poset and $\mathbf{Y} \subseteq \mathbf{X}$. Suppose F is a function which assigns to each $y \in \mathbf{Y}$ an interval (alternately, a unit interval) such that $y_1, y_2 \in \mathbf{Y}$ and $y_1 < y_2$ imply $F(y_1) < F(y_2)$. Then F can be extended to $\mathbf{X}$, i.e. for each $x \in \mathbf{X} - \mathbf{Y}$ an interval (alternately, a unit interval) $F(x)$ may be chosen so that $x_1, x_2 \in \mathbf{X}$ and $x_1 < x_2$ imply $F(x_1) < F(x_2)$.*

We will find it convenient to adopt the convention of saying that the dimension, weak dimension, interval dimension, and semiorder dimension of a one point poset is zero.

3. A Hiraguchi theorem for interval dimension

In 1955 Hiraguchi [9] proved that $\dim(\mathbf{X}) \leq \frac{1}{2}|\mathbf{X}|$ for all $\mathbf{X}$ with $|\mathbf{X}| \geq 4$. A simple proof of Hiraguchi's theorem may be found in [15]. It has been conjectured that an even stronger result holds, namely that every poset of three or more points contains a pair of points whose removal lowers the dimension at most one. There are many conditions under which such a pair is known to exist. As an example, Hiraguchi [9] proved that if a and b are incomparable, a is maximal, and b is minimal, then $\dim(\mathbf{X}) \leq 1 + \dim(\mathbf{X} - \{a, b\})$. To extend this notion, we say that (a, b) satisfies property M if aIb but $z > a$ implies $z > b$ and $z < b$ implies $z < a$. In a finite poset which is not a chain, an ordered pair (a, b) satisfying property M exists. To see that this is the case, choose an incomparable pair (a, b) with the cardinality of $\{z : z \geq a\} \cup \{z : z \leq b\}$ as small as possible. We conjecture that for any poset $\mathbf{X}$, the removal of a pair satisfying property M reduces the dimension at most 1. In support of this conjecture we offer the following theorem.

Theorem 3.1. *Suppose* $\mathbf{X}$ *is a poset and* (a, b) *satisfies property* M; *then the removal of* a *and* b *reduces the interval dimension of* $\mathbf{X}$ *at most one, i.e.* $\mathrm{Idim}(\mathbf{X}) \leq 1 + \mathrm{Idim}(\mathbf{X} - \{a, b\})$.

Proof. Suppose $\mathrm{Idim}(\mathbf{X} - \{a, b\}) = t$; then choose an interval coordinatization F of $\mathbf{X} - \{a, b\}$ of length t. We use this coordinatization to construct an interval coordinatization G of $\mathbf{X}$ of length $t + 1$.

For each $x \in \mathbf{X} - \{a, b\}$, let $G(x)(i) = F(x)(i)$ for all $i \leq t$. By the interpolation lemma, we may choose for each $i \leq t$ intervals $G(a)(i)$ and $G(b)(i)$ so that $x, y \in \mathbf{X}$ and $x < y$ imply $G(x)(i) < G(y)(i)$.

Now partition $\mathbf{X} - \{a, b\}$ into five subposets:

$$\mathbf{X}_1 = \{x \in \mathbf{X} : x > a\}, \quad \mathbf{X}_2 = \{x \in \mathbf{X} : x I a, x > b\}, \quad \mathbf{X}_3 = \{x \in \mathbf{X} : x I a, x I b\},$$

$$\mathbf{X}_4 = \{x \in \mathbf{X} : x < a, x I b\}, \quad \mathbf{X}_5 = \{x \in \mathbf{X} : x < b\}.$$

Let $m_j = |\mathbf{X}_j|$ for each $j \leq 5$. Then define the following open intervals:

$$U_1 = (2, \infty), \quad U_2 = (1, 2), \quad U_3 = (-1, 1),$$

$$U_4 = (-2, -1), \quad U_5 = (-\infty, -2).$$

For each $j \leq 5$, choose m_j points $P_{j1} < P_{j2} < \ldots < P_{jm_j}$ from U_j and let $x_{j1} < x_{j2} < \ldots < x_{jm_j}$ be any linear extension of the subposet $\mathbf{X}_j$.

To complete the coordinatization let $G(a)(t + 1) = [-1, 2]$, $g(b)(t + 1) = [-2, 1]$, and $G(x_{ji})(t + 1) = P_{ji}$ for $j = 1, 2, 3, 4, 5$ and $i = 1, 2, \ldots, m_j$. It is easy to verify that G is an interval coordinatization of $\mathbf{X}$ of length $t + 1$ and it follows that $\mathrm{Idim}(\mathbf{X}) \leq t + 1$.

We note that the conjecture presented in this section has not been settled for semiorder dimension.

4. Other inequalities

We begin the section by proving some removal theorems analogous to the results appearing in [2, 4, 15].

Theorem 4.1. *Let* $x \in \mathbf{X}$ *and let* C *be a chain in* $\mathbf{X}$. *Then*:

(1) $\mathrm{Idim}(\mathbf{X}) \leq 1 + \mathrm{Idim}(\mathbf{X} - x)$.

(2) $\mathrm{Sdim}(\mathbf{X}) \leq 1 + \mathrm{Sdim}(\mathbf{X} - x)$.

(3) $\mathrm{Idim}(\mathbf{X}) \leq 2 + \mathrm{Idim}(\mathbf{X} - C)$.

(4) $\mathrm{Sdim}(\mathbf{X}) \leq 2 + \mathrm{Sdim}(\mathbf{X} - C)$.

Proof. To prove statement (1) (statement (2)), suppose $\mathrm{Idim}(X - x) = t$ ($\mathrm{Sdim}(X - x) = t$). Then choose an (unit) interval coordinatization F of $\mathbf{X} - x$ of length t. For each $y \in \mathbf{X} - x$ and each $i \leq t - 1$ let $G(y)(i) = F(x)(i)$. For each $i \leq t - 1$, extend G to all of $\mathbf{X}$ by the interpolation lemma. Then define

$$\mathbf{X}_1 = \{y \in X : y > x\}, \qquad \mathbf{X}_2 = \{y \in X : yIx\}, \qquad \mathbf{X}_3 = \{y \in I : y < x\}$$

and consider $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_1 \cup \mathbf{X}_2$, and $\mathbf{X}_2 \cup \mathbf{X}_3$ as posets by taking as partial ordering $a < b$ iff $F(a)(t) \lhd F(b)(t)$. Then let H and K be unit interval coordinatizations of the interval orders (semiorders) $\mathbf{X}_3 \oplus \{x\} \oplus (\mathbf{X}_1 \cup \mathbf{X}_2)$ and $\mathbf{X}_3 \cup \mathbf{X}_2 \oplus \{x\} \oplus \mathbf{X}_1$ respectively.

For each $y \in \mathbf{X}$ let $G(y)(t) = H(y)$ and $G(y)(t + 1) = K(y)$. It is easy to verify that G is a (unit) interval coordinatization of $\mathbf{X}$ of length $t + 1$ and thus $\mathrm{Idim}(\mathbf{X}) \leq t + 1$ ($\mathrm{Sdim}(\mathbf{X}) \leq t + 1$).

To prove statement (3) (statement (4)) let F be any (unit) interval coordinatization of $\mathbf{X} - C$ of length t. For each $y \in \mathbf{X} - C$ and each $i \leq t$ let $G(y)(i) = F(y)(i)$. For each $i \leq t$, use the interpolation lemma to extend G to all of $\mathbf{X}$. Now let (X, L_1) and (X, L_2) be upper and lower extensions of the chain C and let H and K be (unit) interval coordinatizations of (X, L_1) and (X, L_2). Then for each $x \in X$, let $G(x)(t + 1) = H(x)$ and $G(x)(t + 2) = K(x)$. It follows that G is an (unit) interval coordinatization of $\mathbf{X}$ of length $t + 2$.

If M denotes the set of maximal elements of a poset X, it is proved in [15] that $\dim(\mathbf{X}) \leq 1 + W(\mathbf{X} - M)$ and a family of posets is constructed to show that this inequality is best possible. It is straightforward to modify the argument in [15] to show that the inequality $\mathrm{Sdim}(\mathbf{X}) \leq 1 + \mathrm{W}(\mathbf{X} - M)$ is also best possible. For interval dimension we have:

Theorem 4.2. $\mathrm{Idim}(\mathbf{X}) \leq \mathrm{W}(\mathbf{X} - M)$.

Proof. Let $\mathrm{W}(\mathbf{X} - M) = t$ and let $C_1 \cup C_2 \cup \ldots \cup C_t$ be a decomposition of $\mathbf{X} - M$ into chains provided by Dilworth's theorem [6]. For each $i \leq t$, let $(X - M, L_i)$ be a lower extension of C_i. Then let $P_i = P \cup L_i$. It follows that each poset (X, P_i) is an interval order and for each $i \leq t$ we choose an interval coordinatization H_i of (X, P_i). Then let $G(x)(i) = H_i(x)$. It follows that G is an interval coordinatization of X of length t.

In [14], the crown $\mathbf{S}_n^k$ is defined and the formula $\dim(\mathbf{S}_n^k) = \{2(n + k)/(k + 2)\}$ is established. We note that the same formula holds for the weak dimension, interval dimension, and semiorder dimension of S_n^k. Thus for $k = 0$, $\mathrm{Idim}(\mathbf{S}_n^0) = n = \mathrm{W}(\mathbf{S}_n^0 - M)$, i.e. the inequality in Theorem 4.1 is best possible. Note that for $n \geq 3$, the poset S_n^0 is isomorphic to the collection of 1-element and $(n - 1)$-element subsets of an n-element set ordered by inclusion.

For a poset $\mathbf{X}$, we define the *breadth* of $\mathbf{X}$, denoted $\mathrm{B}(\mathbf{X})$, to be 1 if X is a chain. If

$\mathbf{X}$ is not a chain but does not contain a subposet isomorphic to $\mathbf{S}_n^0$ for $n \geq 3$, we define the breadth of $\mathbf{X}$ to be 2. If $\mathbf{X}$ contains a subposet isomorphic to S_n^0 for some $n \geq 3$, we define $\mathrm{B}(\mathbf{X})$ to be the largest integer m such that $S_m^0 \subseteq \mathbf{X}$. If $\mathbf{X}$ is a lattice, this definition agrees with the usual definition of breadth for a lattice. It follows then that the inequalities $\mathrm{B}(\mathbf{X}) \leq \dim(\mathbf{X}) \leq \mathrm{W}(\mathbf{X})$ hold and are best possible.

If A is an antichain of $\mathbf{X}$, it is proved in [15] that the inequality $\dim(\mathbf{X}) \leq 2\mathrm{W}(\mathbf{X} - A) + 1$ always holds and a family of posets is produced to show that the result is best possible. The arguments in this paper can be modified to show that the inequality $\mathrm{Sdim}(\mathbf{X}) \leq 2\mathrm{W}(\mathbf{X} - A) + 1$ is also best possible. Once again for interval dimension the result is slightly different.

Theorem 4.3. $\mathrm{Idim}(\mathbf{X}) \leq 2\mathrm{W}(\mathbf{X} - A) - 1$.

Proof. If $\mathrm{W}(\mathbf{X} - A) = 1$, then $\mathbf{X}$ does not contain $\mathbf{2} + \mathbf{2}$ and is therefore an interval order. For $\mathrm{W}(\mathbf{X} - A) = t > 1$, the theorem follows from induction on t and statement (3) of Theorem 3.1.

We note that it is not known whether or not the result of Theorem 4.3 is best possible.

5. Inequalities involving length

In this section we derive some inequalities which show the relationship between $\mathrm{Idim}(\mathbf{X})$, $\mathrm{Sdim}(\mathbf{X})$, and $\mathrm{L}(\mathbf{X})$.

Suppose $\mathbf{X}$ is a poset whose length is n. Let A_n be the set of maximal elements of $\mathbf{X}$. If A_{k+1} has been defined, then let A_k be the maximal elements of $\mathbf{X} - A_n - A_{n-1} - \dots - A_{k+1}$. This construction partitions $\mathbf{X}$ into n antichains. For each $x \in \mathbf{X}$ we define the level of x, denoted $\mathrm{L}(x)$ as the unique i for which $x \in A_i$. We note that $x > y$ in P implies $\mathrm{L}(x) \geq \mathrm{L}(y)$. We also note that in a semiorder $\mathrm{L}(x) \geq \mathrm{L}(y) + 2$ implies $x > y$ in P.

The results developed in this section are motivated by the following pair of theorems discovered by Rabinovitch [11].

Theorem 5.1. *If* $\mathbf{X}$ *is a semiorder, then* $\dim(\mathbf{X}) \leq 3$.

Proof. Let $A = \{x \in \mathbf{X} : \mathrm{L}(x) \text{ is even}\}$ and let $B = \{x \in \mathbf{X} : \mathrm{L}(x) \text{ is odd}\}$. Then A and B are disjoint subsets of an interval order and in view of Lemma 1.3, there exists an injection L_1 of A over B and an injection L_2 of B over A. Now suppose $\mathrm{L}(X) = n$ and let $X = A_1 \cup A_2 \cup \dots \cup A_n$ be the natural partition of X into antichains. For each i with $1 \leq i \leq n$, let A'_i be the dual of the restriction of L_i to A_i. Then define $L_3 = A'_n \oplus A'_{n-1} \oplus \dots \oplus A'_3 \oplus A'_2 \oplus A'_1$. It follows immediately that $L_1 \cap L_2 \cap L_3 = P$ and thus $\dim(\mathbf{X}) \leq 3$.

The inequality of Theorem 5.1 is best possible as there exist three dimensional semiorders. Rabinovitch [11] has also given a characterization theorem for such posets.

Theorem 5.2. *There exists a function $f(x)$ so that if* **X** *is an interval order of length n, then* $\dim(\mathbf{X}) \leq f(n)$.

We do not include a proof of Theorem 5.2 although the reader may easily establish Rabinovitch's original result that the dimension of an interval order of length n is at most $\{1 + \log_2 n\}$. We note that this bound was improved by the authors and Rabinovitch [3]. Subsequently Trotter [18] proved that there is a positive constant c so that an interval order of length n has dimension at most $c \log_2 \log_2 n$. Trotter also proved that for each $n \geq 1$, there exists an interval order of length n whose dimension is at least $2 \log_2 \log_2 n$. In view of the intimate connection between Trotter's bounds and Ramsey theory, it is not likely that a computation of the best possible value of $f(n)$ in Theorem 5.2 is feasible.

It is possible, however, to extend Rabinovitch's theorems as follows:

Theorem 5.3. *If* $\mathrm{Sdim}(\mathbf{X}) \leq t$, *then* $\dim(\mathbf{X}) \leq 3t$.

Proof. Suppose that $\mathrm{Sdim}(\mathbf{X}) = t$ and let F be a unit interval coordinatization of length t of **X**. For each $i \leq t$, let (X_i, P_i) be the poset defined by $x < y$ in P_i iff $F(x)(i) < F(y)(i)$. Then each poset (X_i, P_i) is a semiorder and we may choose linear orders L_{i1}, L_{i2} and L_{i3} so that $P_i = L_{i1} \cap L_{i2} \cap L_{i3}$. Since $P = P_1 \cap P_2 \cap \ldots \cap P_t$, it follows that $\dim(\mathbf{X}) \leq 3t$.

Theorem 5.4. *There exists a function $g(n, t)$ so that if X is a poset of length n and interval dimension t, then* $\dim(\mathbf{X}) \leq g(n, t)$.

The proof of Theorem 5.4 is quite similar to that of Theorem 5.3. As we shall see in the next section, determining the best possible values of the function g is a hopelessly difficult problem.

6. Cartesian products

One of the well known elementary inequalities for posets is $\dim(\mathbf{X} \times \mathbf{Y}) \leq \dim(\mathbf{X}) + \dim(\mathbf{Y})$. Not so well known is the following result (see Exercise 7, page 101 of Birkhoff [1]).

Theorem 6.1. *If* **X** *and* **Y** *are posets with distinct universal bounds, then* $\dim(\mathbf{X} \times \mathbf{Y}) = \dim(\mathbf{X}) + \dim(\mathbf{Y})$ *and* $\mathbf{B}(\mathbf{X} \times \mathbf{Y}) = \mathbf{B}(\mathbf{X}) + \mathbf{B}(\mathbf{Y})$.

We will next derive a variant of Theorem 6.1 for interval dimension.

Theorem 6.2. *Let* $\mathbf{X} = (X, P)$ *and* $\mathbf{Y} = (Y, Q)$ *be posets with distinct universal bounds. If neither* $\mathbf{X}$ *nor* $\mathbf{Y}$ *is a chain, then* $\text{Idim}(\mathbf{X} \times \mathbf{Y}) \geq \dim(\mathbf{X}) + \dim(\mathbf{Y})$.

Proof. Suppose $\text{Idim}(\mathbf{X} \times \mathbf{Y}) = t$ and let F be an interval coordinatization of $\mathbf{X} \times \mathbf{Y}$ of length t. We denote the universal bounds of $\mathbf{X}$ by 0 and 1; similarly we denote the universal bounds of $\mathbf{Y}$ by 0′ and 1′. Then we may assume without loss of generality that there exists an integer s with $1 \leq s \leq t$ so that the right end point of $F((0, 1'))(i)$ is greater than or equal to the right end point of $F((1, 0'))(i)$ iff $1 \leq i \leq s$. We now show that $s \geq \dim(\mathbf{Y})$.

Let i be an integer with $1 \leq i \leq s$; then define a subset $S_i \subseteq \mathcal{I}_Q$ by $S_i = \{(y_1, y_2) \in \mathcal{I}_Q$: right end point of $F((0, y_2))(i)$ is at least as large as the left end point of $F((1, y_1))(i)\}$. Suppose that some S_i contains a strong TM-cycle $\{(a_j, b_j): 1 \leq j \leq m\}$ of length m. Then for each j with $1 \leq j \leq m$, we know that the right end point of $F((0, b_j))(i)$ is at least as large in $\mathbf{R}$ as the left end point of $F((1, a_j))(i)$. Since $b_j < a_{j+1}$ in Q, it follows that $(0, b_j) < (1, a_{j+1})$ in $\mathbf{X} \times \mathbf{Y}$ and therefore $F((0, b_j))(i) \lhd F((1, a_{j+1}))(i)$. It follows that for each j with $1 \leq j \leq m$, the left end point of $F((1, a_{j+1}))(i)$ is larger than the left end point of $F((1, a_j))(i)$. Clearly, this is not possible. We may conclude that for each i with $1 \leq i \leq s$, the relation $Q_i = \overline{Q \cup S_i}$ is a partial order on Y. Now for each i with $1 \leq i \leq s$, let M_i be a linear extension of Q_i. We next show that $Q = L_1 \cap L_2 \cap \ldots \cap L_s$.

Suppose $(y_1, y_2) \in \mathcal{I}_Q$ but that $(y_1, y_2) \notin S_i$ for each i with $1 \leq i \leq s$. Then it follows that $F((0, y_2))(i) \lhd F((1, y_1))(i)$ for all i with $1 \leq i \leq s$. Now $(0, y_2)\, I\, (1, y_1)$ in $\mathbf{X} \times \mathbf{Y}$ but $(0, y_2) < (0, 1')$ and $(1, 0') < (1, y_1)$. Furthermore the right end point of $F((1, 0'))(i)$ is larger than the right end point of $F((0, 1'))(i)$ for each i with $s < i \leq t$. This in turn implies that $F((0, y_2))(i) \lhd F((1, y_1))(i)$ for all i with $1 \leq i \leq t$. The contradiction shows that there must exist some i with $1 \leq i \leq s$ so that $(y_1, y_2) \in S_i$. It follows immediately that $Q = L_1 \cap L_2 \cap \ldots \cap L_s$ and therefore $s \geq \dim(\mathbf{Y})$.

The argument that $t - s \geq \dim(\mathbf{X})$ is dual and is therefore omitted.

We note that it follows immediately from Theorem 2.2 that $\text{Idim}(\mathbf{2} \times \mathbf{2}) = \text{Idim}(\mathbf{2} \times \mathbf{3}) = 1$ while $\text{Idim}(\mathbf{2} \times \mathbf{4}) = \text{Idim}(\mathbf{3} \times \mathbf{3}) = 2$. It is then trivial to modify the proof of Theorem 6.2. to obtain the following corollaries.

Corollary 6.3. $\text{Idim}(\mathbf{X} \times \mathbf{2}) \geq \dim(\mathbf{X})$ *for every poset* X.

Corollary 6.4. *Let* $\mathbf{X}$ *be a poset with distinct universal bounds. If* $\mathbf{X}$ *contains at least 3 points, then* $\text{Idim}(\mathbf{X} \times \mathbf{3}) = 1 + \dim(\mathbf{X})$.

Corollary 6.5. *Let* $\mathbf{X}$ *be a poset with distinct universal bounds. If* $\mathbf{X}$ *contains at least 4 points, then* $\text{Idim}(\mathbf{X} \times \mathbf{2}) \geq 1 + \dim(\mathbf{X})$.

Theorem 6.2 and the corollaries reveal that interval dimension does not behave well with respect to the algebraic operation of cartesian product. Intuitively, we would prefer that if **X** and **Y** are relatively simple posets, then each of the posets **X** + **Y**, **X** ⊕ **Y**, and **X** × **Y** should also be relatively simple. However, we see that the cartesian product of an interval order and a 2-element chain can have arbitrarily large interval dimension. This obvious drawback for interval dimension must be weighed against its many advantages. For example, Theorems 2.2 and 2.4 show that the inequality Idim(**X**) ≤ ½|**X**| holds for all **X**. Furthermore we invite the reader to compare the characterization of this inequality for interval dimension [21] with the corresponding result for dimension [4]. Moreover, the removal theorems for interval dimension are much more elegant than the corresponding theorems for dimension.

7. Concluding remarks and open problems

Originally, the authors' motivation for studying such concepts as interval dimension and semiorder dimension came from our desire to merge the applications of semiorders and interval orders in the theory of measurement with the concept of dimension theory. However, certain unexpected mathematical benefits have surfaced. First, the concept of interval dimension has proved to be a key link between a number of combinatorial problems. In [20] Trotter and Moore use interval dimension to relate problems involving posets to important characterization problems involving circular arc graphs, comparability graphs, and planar lattices. Furthermore, the concept of interval dimension has been used in the construction of irreducible posets of arbitrary cardinality [17].

Finally we note that the results in the preceding section impart further significance to determining whether or not for each pair of positive integers m and n with $m \geqslant n$, there exists posets **X** and **Y** with dim(**X**) = m, dim(**Y**) = n and dim(**X** × **Y**) = m.

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