**Journal Title:** The Theory and Applications of Graphs

**Volume:**
**Issue:**
**Month/Year:** 1981
**Pages:** 397 - 407

**Article Author:** Hopkins and Trotter

**Article Title:** A bound on the interval number of a complete multipartite graph



William Trotter (wt48)
School of Mathematics
Georgia Tech
Atlanta, GA 30332

**Faculty**
**Math**

which joining sets of (and diverting) paths, r 1977), 18-28.

n which players of gger game, *Scientific* .

Wesley, Reading, (1969).

of the acquaintance graph,

wright, *Structural Models: Directed Graphs*, Wiley,

. Wormald, Isomorphic s, *Trans. Amer. Math. Soc.*,

Solution of Irving's to appear.

roblem and the Zarankiewicz 78), 13-26.

al logic, *Proc. London*

g strategy for the second *Tenth S.E. Conference on h Theory*, Utilitas,

*Recreational Math.*, 2


# A Bound on the Interval Number of a Complete Multipartite Graph


**L.B. HOPKINS**

**W.T. TROTTER***



ABSTRACT

The *interval number* of a graph $G$ denoted $i(G)$, is the least positive integer $t$ for which $G$ is the intersection graph of a family of sets each of which is the union of $t$ pairwise disjoint intervals of the real line. For example, a graph $G$ is an interval graph if and only if $i(G) = 1$, while $i(C_n) = 2$ for all $n \geq 4$. Griggs showed that the maximum value of the interval number of a graph on $n$ vertices is $\lceil (n+1)/4 \rceil$ and Trotter and Harary showed that the interval number of the complete bipartite graph $K(n_1,n_2)$ is given by the formula $i(K(n_1,n_2)) = \lceil (n_1n_2+1)/(n_1+n_2) \rceil$. Several researchers have been investigating the problem of determining the interval number of complete multipartite graphs, and it was conjectured that the interval number of the complete multipartite graph $K(n_1,n_2,n_3,\ldots,n_p)$, where $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_p$ and $p \geq 3$, equals the interval number of the complete bipartite graph $K(n_1,n_2)$. In support of this conjecture, Matthews proved that for every $p \geq 3$, if $n_1 = n_2 = n_3 = \ldots = n_p$, then $i(K(n_1,n_2,\ldots,n_p)) = i(K(n_1,n_2))$. However, D. West disproved the conjecture by showing that for each $n \geq 3$, there exists a constant $c_n$ so that if $n_1 = n^2 - n - 1$, $n_2 = n_3 = \ldots = n_p = n$, and $p \geq c_n$, then $i(K(n_1,n_2,n_3,\ldots,n_p)) = 1 +$

$i(K(n_1,n_2))$ . In view of West's counterexample, it was suggested that the interval number of a complete multipartite graph might exceed the interval number of the bipartite graph, formed by the largest two parts, by an arbitrarily large amount. In this paper, we prove to the contrary that $i(K(n_1,n_2,n_3,\ldots,n_p)) \leq 1 + i(K(n_1,n_2))$ for all $p, n_1, n_2, \ldots, n_p$ with $p \geq 3$ and $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_p$ .

## 1. Introduction.

In recent years, there has been considerable interest in generalizations of interval graphs. Much of the research is motivated by the wide range of interpretations which may be given to optimization and extremal problems involving interval graphs. In this paper, we consider the subject of t-interval graphs. For a positive integer $t$ , we represent a graph as the intersection graph of a family of sets each of which is the union of $t$ pairwise disjoint intervals of the real line. Among the several extremal problems involving t-interval graphs, we will be concerned with minimizing $t$ for a given graph or class of graphs. If we view a t-interval graph as a work schedule permitting cooperation between certain specified components of the work force while safeguarding against interference between other components, then the minimization of $t$ yields a schedule in which each component has relatively few work periods. Consequently, the inherent inefficiency of starting up and closing down unnecessary work periods of short duration and limited productivity is avoided.

Among the classes of graphs for which this extremal problem is quite natural is the class of complete bipartite graphs where the work force is subdivided into two units with no interference permitted between any two components in the same unit, but cooperation required between any two components from different



units. In this paper, we will
to complete multipartite graph
problem in the multipartite ca
from the bipartite graph.

2. Notation and Terminology.

Trotter and Harary [4] de
of a graph $G$ as a function
$x \in G$ a sequence $F(x)(1)$ , F
intervals of the real line $\mathbb{R}$
distinct vertices, we have $x$
if there exists a pair of inte
that $F(x)(i) \cap F(y)(j) \neq \emptyset$ .
denoted $i(G)$ , is then defin
for which $G$ has a t-interval
$i(G)$ is the least integer $t$
graph of a family of sets each
$t$ closed intervals of $\mathbb{R}$ . I
interval graph if and only if

We will find it convenien
(degenerate) closed interval a
vention in specifying an inter
For example, Figure 1 provid
of the graph $G$ . Note that
$i(G) = 2$ .

Furthermore, we will dele
representation, all isolated i
may be simplified as in Figure

Throughout this paper, we
shown in Figures 1 and 2 to il
Intervals will be spread out
reader should bear in mind tha

units. In this paper, we will discuss the natural generalization to complete multipartite graphs and will show that the extremal problem in the multipartite case does not differ substantially from the bipartite graph.

2. Notation and Terminology.

Trotter and Harary [4] defined a t-*interval representation* of a graph $G$ as a function $F$ which assigns to each vertex $x \in G$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(t)$ of closed intervals of the real line $\mathbb{R}$ so that for every pair $x, y$ of distinct vertices, we have $x$ adjacent to $y$ in $G$ if and only if there exists a pair of integers $i, j$ with $1 \leq i, j \leq n$ so that $F(x)(i) \cap F(y)(j) \neq \emptyset$. The *interval number* of a graph $G$, denoted $i(G)$, is then defined as the least positive integer $t$ for which $G$ has a t-interval representation. Alternately, $i(G)$ is the least integer $t$ for which $G$ is the intersection graph of a family of sets each of which is the union of at most $t$ closed intervals of $\mathbb{R}$. In particular, a graph $G$ is an interval graph if and only if $i(G) = 1$.

We will find it convenient to consider a point as a (degenerate) closed interval and will frequently use this convention in specifying an interval representation of a graph. For example, Figure 1 provides a 2-interval representation of the graph $G$. Note that $G$ is not an interval graph so $i(G) = 2$.

Furthermore, we will delete from Figure 1 for an interval representation, all isolated intervals and points. So Figure 1 may be simplified as in Figure 2.

Throughout this paper, we will use diagrams similar to those shown in Figures 1 and 2 to illustrate interval representations. Intervals will be spread out vertically for clarity but the reader should bear in mind that all intervals are to be projected

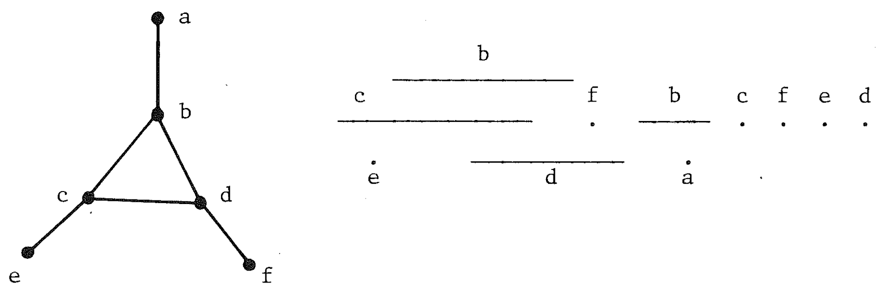



Figure 1.

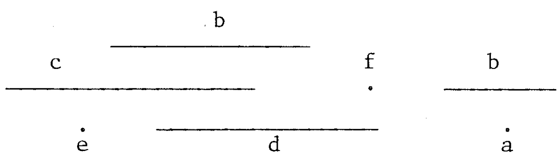



Figure 2.

onto a single horizontal line.

We now present a brief summary of recent research involving interval numbers. We begin with the following elementary result due to Trotter and Harary [4].

*Theorem 1* [*4*]. If $T$ is a tree, then $i(T) \leq 2$.

J. Griggs [2] has established the following upper bound on the interval number as a function of the order of the graph.

*Theorem 2* [*2*]. If $G$ is a graph on $n$ vertices, then $i(G) \leq \lceil (n+1)/4 \rceil$.



J. Griggs and D. West [1]
bound on $i(G)$ as a function o
in $G$.

*Theorem 3* [*1*]. If $\Delta$ is the
then $i(G) \leq \lceil (\Delta+1)/2 \rceil$.

Griggs and West [1] showe
triangle-free, then equality h
result follows as an immediate

*Corollary 4* [*1*]. For each $n$
n-cube $Q_n$ is given by $i(Q_n)$

Trotter and Harary [4] de
number of a complete bipartite

*Theorem 4* [*4*]. The interval n
graph $K(m,n)$ is given by:

$$i(K(m,n)) =$$

3. <u>Interval Numbers of Compl</u>

In the remainder of this
the computation of the interva
partite graph $K(n_1,n_2,\ldots,n_p)$
will require $n_1 \geq n_2 \geq \ldots \geq$
$i(n_1,n_2,\ldots,n_p)$ denote the in
note that $i(n_1,n_2) = \lceil (n_1 n_2 +$
$K(n_1,n_2)$ is an induced subgra
have $i(n_1,n_2) \leq i(n_1,n_2,\ldots,n$
$i(n_1,n_2,\ldots,n_p)$ never exceeds
construction due to M. Matthew
$K(n,n)$ be $A \cup B$ where $A =$
with each $a_i$ adjacent to eve

First let $n$ be even, sa

J. Griggs and D. West [1] have also established an upper bound on $i(G)$ as a function of the maximum degree of a vertex in $G$.

*Theorem 3* [*1*]. If $\Delta$ is the maximum degree of a vertex in $G$, then $i(G) \leq \lceil(\Delta+1)/2\rceil$.

Griggs and West [1] showed that if $G$ is regular and triangle-free, then equality holds in Theorem 3. The following result follows as an immediate corollary.

*Corollary 4* [*1*]. For each $n \geq 1$, the interval number of the n-cube $Q_n$ is given by $i(Q_n) = \lceil(n+1)/2\rceil$.

Trotter and Harary [4] developed a formula for the interval number of a complete bipartite graph.

*Theorem 4* [*4*]. The interval number of the complete bipartite graph $K(m,n)$ is given by:

$$i(K(m,n)) = \lceil(mn+1)/(m+n)\rceil .$$

## 3. Interval Numbers of Complete Multipartite Graphs

In the remainder of this paper, we will be concerned with the computation of the interval number of a complete multipartite graph $K(n_1,n_2,\ldots,n_p)$ where $p \geq 2$. By convention, we will require $n_1 \geq n_2 \geq \ldots \geq n_p$. For simplicity, we let $i(n_1,n_2,\ldots,n_p)$ denote the interval number of $K(n_1,n_2,\ldots,n_p)$; note that $i(n_1,n_2) = \lceil(n_1n_2+1)/(n_1+n_2)\rceil$ by Theorem 4. Since $K(n_1,n_2)$ is an induced subgraph of $K(n_1,n_2,\ldots,n_p)$, we always have $i(n_1,n_2) \leq i(n_1,n_2,\ldots,n_p)$. We now proceed to show that $i(n_1,n_2,\ldots,n_p)$ never exceeds $n_2$. We begin by presenting a construction due to M. Matthews [3]. We let the vertex set of $K(n,n)$ be $A \cup B$ where $A = \{a_1,a_2,\ldots,a_n\}$, $B = \{b_1,b_2,\ldots,b_n\}$ with each $a_i$ adjacent to every $b_j$.

First let $n$ be even, say $n = 2r$. Then the following diagram provides an $r+1$ -interval representation of $K(n,n)$ .

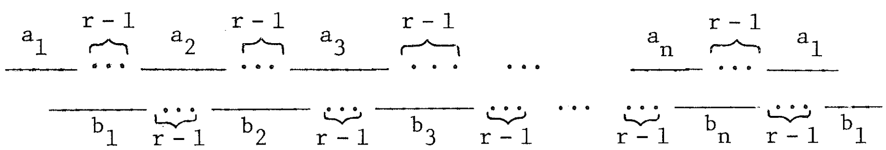



Figure 3.

In the gap between $a_i$ and $a_{i+1}$ (cyclically), occur the $r-1$ points corresponding to $a_{i+2}, a_{i+3}, \ldots, a_{i+r}$ (cyclically). Similarly, the gap between $b_i$ and $b_{i+1}$ contains points corresponding to $b_{i+2}, b_{i+3}, \ldots, b_{i+r}$ . Here is a diagram when $r = 3$ . For simplicity, only the subscripts are given.

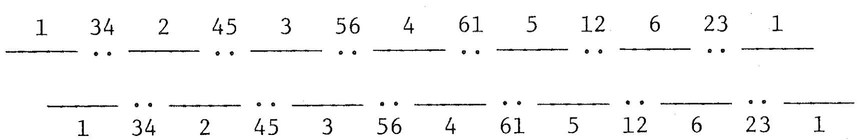



Figure 4 .

We shall continue to use the convention followed in Figures 3 and 4 for bipartite and multipartite graphs i.e., the diagram will be presented in "levels" with all intervals occurring in the same level corresponding to vertices in the same part.

The reader is encouraged to compare this example with the construction given by Trotter and Harary [4] for a 4-representation of $K_{6,6}$ . The advantage of Matthews' construction is that it can easily be extended to multipartite graphs. It suffices to add additional "levels" to the diagram following the same intersection pattern as determined by the first two. For example, here is a 3-representation (with labels deleted) of $K(6,6,6)$ .

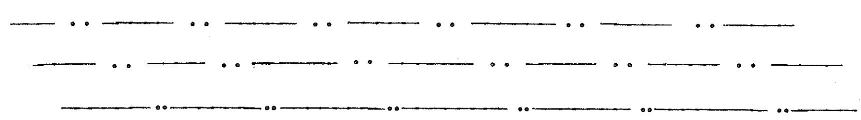


Figure 5.

More generally, it is easy to see that when $n = 2r$ , this construction produces for each $p \geq 2$ , an $r + 1$ - interval representation of $K(n_1,n_2,\ldots,n_p)$ where $n_1 = n_2 = \ldots = n_p = n$ . It follows that when $n = 2r$ , we have $r + 1 = \left\lceil \frac{4r^2+1}{4r} \right\rceil = i(n,n) \leq i(n_1,n_2,\ldots,n_p) \leq r + 1$ , and thus $i(n_1,n_2,\ldots,n_p) = r + 1$ .

When $n$ is odd, say $n = 2r + 1$ then we construct an $r + 2$ -interval representation of $K(n,n)$ using the same scheme as above. For example, here is the diagram for $K(5,5)$

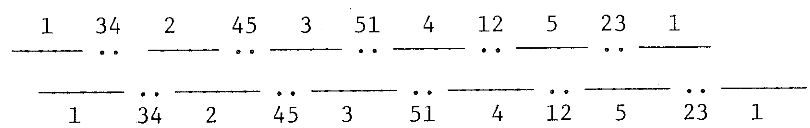



Figure 6.

Reading the diagram from left to right, we then remove from each level the first occurrence of 1 as a point in a gap. The resulting diagram is an $r + 1$ - interval representation of

$K(2r+1, 2r+1)$ .

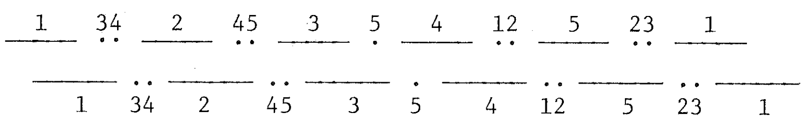



Figure 7.

As before, this construction is easily extended to show that whenever $p \geq 2$ and $n = 2r+1 = n_1 = n_2 = \ldots = n_p$ , then $r+1 = i(n,n) = i(n_1, n_2, \ldots, n_p)$ . We have then established the following result of Mattews [3].

*Theorem 5.* For every $p \geq 2$ and every $n \geq 1$ , if $n_1 = n_2 = \ldots = n_p = n$ , then $i(n_1, n_2, \ldots, n_p) = i(n_1, n_2) = \lceil (n^2+1)/2n \rceil = \lceil (n+1)/2 \rceil$ .

From Theorem 5 we obtain the following upper bound.

*Corollary 6.* If $p \geq 2$ and $n_1 \geq n_2 \geq \ldots \geq n_p$ , then $i(n_1, n_2, \ldots, n_p) \leq n_2$ .

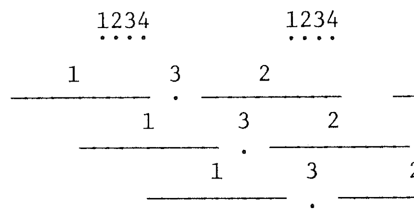
*Proof.* It suffices to establish the result when $p \geq 3$ and $n_2 = n_3 = \ldots = n_p$ . Set $n = n_2$ and then choose a $\lceil (n+1)/2 \rceil$ - interval representation of the complete $p-1$ partite graph $K(n,n,n,\ldots,n)$ as provided in the preceding theorem.

We then observe that for each $i$ with $1 \leq i \leq n$ , there are $p-1$ intervals, one from each level of the diagram, each of which has label $i$ so that the intersection of these $p-1$ intervals is a nondegenerate interval. We may then insert in each of these intervals, $n_1$ points - one for each element in the part of size $n_1$ . The resulting diagram is a $n_2$ - interval



representation of $K(n_1, n_2, \ldots$

We illustrate this result




Fi

For the remainder of the
conventions. We partition the
- partite graph $K(m, n_1, n_2, \ldots,$
$n_p$ and $p \geq 1$ , into the sub
$|A| = m$ and $|B_i| = n_i$ for
vertices in $A$ with the symbo
$i$ , we label the vertices in
$b_{i1}, b_{i2}, \ldots, b_{in_i}$ . When p
representation, we will prese
intervals (or points) correspo
in the highest level which we
downwards, the intervals (or p
in $B_i$ will be displayed in l

When $m \geq n \geq 1$ and $p \geq$
complete $p+1$ - partite grap
$i(m, n \cdot p)$ denote the interval
define $i(m, n \cdot \infty) = \sup\{i(m,n$
result then follows trivially

*Theorem 7.* For every $m, n$
$i(m, n \cdot \infty) \leq n$ .

We next describe a const

representation of $K(n_1,n_2,\ldots,n_p)$ . ∎

We illustrate this result for a diagram for $K(4,3,3,3)$ .

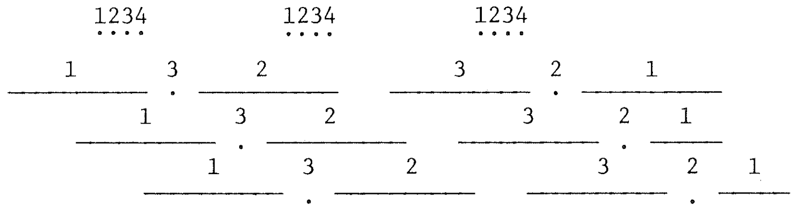



Figure 8.

For the remainder of the paper, we will adopt the following conventions. We partition the vertex set of the complete $p+1$ -partite graph $K(m,n_1,n_2,\ldots,n_p)$ , where $m \geq n_1 \geq n_2 \geq \ldots \geq n_p$ and $p \geq 1$ , into the subsets $A, B_1, B_2, \ldots, B_p$ where $|A| = m$ and $|B_i| = n_i$ for $i = 1, 2, \ldots, p$ . We label the vertices in $A$ with the symbols $a_1, a_2, \ldots, a_m$ . For each $i$ , we label the vertices in $B_i$ with the symbols $b_{i1}, b_{i2}, \ldots, b_{in_i}$ . When providing a diagram for an interval representation, we will present the intervals in levels. The intervals (or points) corresponding to vertices in $A$ will be in the highest level which we call level zero. Then proceeding downwards, the intervals (or points) corresponding to vertices in $B_i$ will be displayed in level $i$ .

When $m \geq n \geq 1$ and $p \geq 1$ , we let $K(m,n \cdot p)$ denote the complete $p+1$ -partite graph $K(m,n,n,\ldots,n)$ and let $i(m,n \cdot p)$ denote the interval number of $K(m,n \cdot p)$ . We then define $i(m,n \cdot \infty) = \sup\{i(m,n \cdot p) : p \geq 1\}$ . The following result then follows trivially.

*Theorem 7.* For every $m, n$ with $m \geq n \geq 1$ , $\left\lceil \frac{mn+1}{m+n} \right\rceil = i(m,n) \leq i(m,n \cdot \infty) \leq n$ .

We next describe a construction generalizing the technique used in Theorem 5. Let $\sigma$ be a sequence (with repetition allowed) of length $\ell$ and let D be a subset of $\{1,2,3,\dots,\ell\}$ with $\{1,\ell\} \subseteq D$. Then we refer to the pair $(\sigma,D)$ as a *sequence with distinguished positions*, or *DP-sequence* for short. To define such a pair, we will find it convenient to list in order the terms of $\sigma$ and underline the distinguished positions, e.g., $(\underline{1},2,3,\underline{2},4,2,\underline{3},\underline{2},4,\underline{1})$. For a DP-sequence $(\sigma,D)$ and an integer $p \geq 1$, we then associate an interval representation having p levels as illustrated below for the DP-sequence given above and $p = 3$.

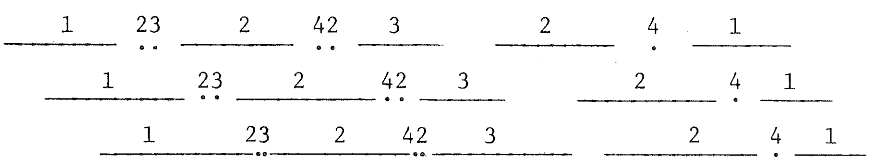



Figure 9.

Suppose we have a DP-sequence $(\sigma,D)$ with the symbols in $\sigma$ selected from $\{1,2,3,\dots,n\}$. Then it is elementary to determine when $(\sigma,D)$ produces an interval representation of $K(n,n,\dots,n)$. (Note that the question does not depend on the number of parts.) For emphasis, we state the characterization of such DP-sequences as a theorem, but we leave it to the reader to supply the straightforward proof.

*Theorem 8.* Let $(\sigma,D)$ be a DP-sequence of length $\ell$ with the symbols in $\sigma$ selected from $\{1,2,3,\dots,n\}$. Also let $D = \{k_1,k_2,\dots,k_d\}$ where $1 = k_1 < k_2 < k_3 < \dots < k_d = \ell$. Then $(\sigma,D)$ produces an interval representation of $K(n,n,n,\dots,n)$ if and only if for every ordered pair $(j_1,j_2)$ from $\{1,2,\dots,n\}$,



there exists an integer $\beta$ wi
one of the following statement

a. $j_1 = \sigma(k_{\beta+1})$ and $j_2$

b. $j_2 = \sigma(k_\beta)$ and $j_1 \in$

An essential feature of t
depends on an Euler circuit in
$n \geq 2$, let $\underset{\sim}{T}(n)$ denote the
tex set $\{1,2,3,\dots,n\}$, i.e.,
$\{(i,j) : 1 \leq i, j \leq n, i \neq j\}$
$\leq \lfloor (n-1)/2 \rfloor$, we let $\underset{\sim}{T}(n,s)$
$\underset{\sim}{T}(n)$ whose edge set is $\{(i,j$
$n + i - s$ (cyclically)$\}$. No
in which each vertex has indeg

It follows easily that $\underset{\sim}{T}$
directed sense). However, in
require an Euler circuit of $\underset{\sim}{T}$
tional property, namely that w
sequence of vertices which beg
consecutive vertices in this s
we proceed to explicitly const
begin with the following eleme

*Lemma 9.* Let $n \geq 2$, $s \geq 1$,
following sequence of vertices
which each $s + 1$ consecutive

$1, s+2, 2, s+3, 3, s$

*Proof.* We note that the
There are then two cases. Whe
directed cycle and the given s
Euler circuit of $\underset{\sim}{T}(n,s)$. Si
$s + 1$ consecutive vertices in

On the other hand, when
but the given sequence is stil

ience (with repetition
a subset of $\{1,2,3,\dots,\ell\}$
the pair $(\sigma,D)$ as a
, or *DP-sequence* for short.
t convenient to list in
the distinguished positions,
DP-sequence $(\sigma,D)$ and an
n interval representation
w for the DP-sequence given

2 4 1
2 4 1
2 4 1

.

$(\sigma,D)$ with the symbols in
en it is elementary to
terval representation of
on does not depend on the
tate the characterization of
e leave it to the reader

nce of length $\ell$ with the
$\dots,n\}$. Also let $D =$
$k_3 < \dots < k_d = \ell$. Then
ation of $K(n,n,n,\dots,n)$ if
$(j_1,j_2)$ from $\{1,2,\dots,n\}$,



there exists an integer $\beta$ with $1 \le \beta < d$ so that at least one of the following statements hold:

a. $j_1 = \sigma(k_{\beta+1})$ and $j_2 \in \{\sigma(k) : k_\beta \le k \le k_{\beta+1}\}$

b. $j_2 = \sigma(k_\beta)$ and $j_1 \in \{\sigma(k) : k_\beta \le k \le k_{\beta+1}\}$.

An essential feature of the construction we are building depends on an Euler circuit in a directed graph. For an integer $n \ge 2$, let $\tilde{T}(n)$ denote the complete directed graph with vertex set $\{1,2,3,\dots,n\}$, i.e., the edge set of $\tilde{T}(n)$ is $\{(i,j) : 1 \le i, j \le n, i \ne j\}$. For an integer $s$ with $1 \le s \le \lfloor (n-1)/2 \rfloor$, we let $\tilde{T}(n,s)$ denote the spanning subgraph of $\tilde{T}(n)$ whose edge set is $\{(i,j) : 1 \le i \le n, i+s+1 \le j \le n+i-s \text{ (cyclically)}\}$. Note that $\tilde{T}(n,s)$ is a regular graph in which each vertex has indegree and outdegree $n-2s$.

It follows easily that $\tilde{T}(n,s)$ has an Euler circuit (in the directed sense). However, in a construction to follow, we shall require an Euler circuit of $\tilde{T}(n,s)$ which satisfies an additional property, namely that when the circuit is specified by a sequence of vertices which begins and ends at $1$, any $s+1$ consecutive vertices in this sequence are distinct. To this end we proceed to explicitly construct such an Euler circuit. We begin with the following elementary result.

*Lemma 9.* Let $n \ge 2$, $s \ge 1$, and $s = \lfloor (n-1)/2 \rfloor$. Then the following sequence of vertices is an Euler circuit of $\tilde{T}(n,s)$ in which each $s+1$ consecutive vertices are distinct:

$$1, s+2, 2, s+3, 3, s+4, \dots, n-1, s, n, s+1, 1.$$

*Proof.* We note that the hypothesis requires that $n \ge 3$. There are then two cases. When $n$ is odd, $\tilde{T}(n,s)$ is a directed cycle and the given sequence is easily seen to be an Euler circuit of $\tilde{T}(n,s)$. Since $s+1 \le n$, we know that any $s+1$ consecutive vertices in the sequence are distinct.

On the other hand, when $n$ is even, $\tilde{T}(n,s)$ has $2n$ edges but the given sequence is still an Euler circuit. A set of $s+1$

consecutive vertices in this sequence has the following form:

$$\{i, i+1, i+2, \ldots, i+s_1-1\} \cup \{i+s+1, i+s+2, \ldots, i+s+s_2\}$$

where $s_1, s_2 > 0$ and $s_1 + s_2 = s + 1$. Since these $s + 1$ integers are distinct, the desired result follows. ■

*Lemma 10.* Let $n \geq 2$ and $1 \leq s < \lfloor (n-1)/2 \rfloor$. Then the sequence: $1, s+2, 2, s+3, 3, \ldots, n-1, s, n, s+1, 1$ traverses a set of $2n$ edges in $\tilde{T}(n,s)$. If these $2n$ edges are removed from $\tilde{T}(n,s)$, then the remaining graph is $\tilde{T}(n,s+1)$.

*Proof.* It suffices to observe that the sequence traverses exactly the edges in the following sets: $\{(i,i+s+1) : 1 \leq i \leq n\} \cup \{(i,i-s) : 1 \leq i \leq n\}$. But this set consists of precisely those edges which belong to $\tilde{T}(n,s)$ but not $\tilde{T}(n,s+1)$. ■

*Lemma 11.* Let $n \geq 2$, $s \geq 1$, and $1 \leq s \leq \lfloor (n-1)/2 \rfloor$. Then $\tilde{T}(n,s)$ has an Euler circuit in which each $s + 1$ consecutive vertices are distinct.

*Proof.* The result follows from Lemma 9 when $s = \lfloor (n-1)/2 \rfloor$. So we may assume that $s < \lfloor (n-1)/2 \rfloor$. We then construct an Euler circuit $\sigma$ by recursively applying Lemma 10. It remains only to show that every set of $s + 1$ vertices in $\sigma$ is distinct. Let $S = \{\sigma(j) : j_0 \leq j \leq j_0 + s\}$ be a set of $s + 1$ consecutive vertices in $\sigma$. Then let $S_1 = \{\sigma(j) : j_0 \leq j \leq j_0 + s,\ j \text{ odd}\}$ and $S_2 = \{\sigma(j) : j_0 \leq j \leq j_0 + s,\ j \text{ even}\}$. Note that $S_1$ is always a set of consecutive integers (cyclically). However, for some values of $j_0$, $S_2$ is a set of consecutive integers (cyclically), and for other values of $j_0$, $S_2$ is "almost" a set of consecutive integers with only a single missing integer preventing it from being a set of consecutive integers.

Suppose first that $j_0$ is odd. If we let $\sigma(j_0) = i$ and $s_1 = |S_1|$, then $s_1 = \lceil (s+1)/2 \rceil$ and $S_1 = \{i, i+1, i+2, \ldots, i+s_1-1\}$. Now let $\sigma(j_0+1) = i + s_3 + 1$



and $s_2 = |S_2|$. Then $s_2 = \lfloor($
and $S_2$ is a subset of the fol
integers $S_2^* = \{i+s_3+1, i+s_3$
$s_1 - 1 < s_3 + 1$ and $s_3 + s_2 +$
$S_2^* = \emptyset$ and thus, the $s + 1$ v

Now consider the case when
$s_1 = |S_1|$; also let $i = \sigma(j_0$
and $S_1 = \{i, i+1, i+2, \ldots, i+$
and $s_2 = |S_2|$. Then $s_2 = \lceil($
and $S_2$ is a subset of the fol
integers $S_2^* = \{i+s_3, i+s_3+1$
$s_3$ and $s_3 + s_2 < n$, it foll
the $s + 1$ consecutive intege
observation, the proof is compl

At the risk of belaboring
sequence determines an Euler c
set of 3 consecutive vertice
4 , 7 , 5 , 8 , 6 , 9 , 7 , 1 , 8 , 2 ,
9 , 6 , 1 , 7 , 2 , 8 , 3 , 9 , 4 , 1 ,

We need one last concept
principal theorem. Let $(\sigma_1, D$
and let $(\sigma_2, D_2)$ be a DP-sequ
$\sigma_1(\ell_1) = \sigma_2(1)$, we define th
denoted $(\sigma_1,D_1) \oplus (\sigma_2,D_2)$,
has length $\ell = \ell_1 + \ell_2 - 1$, $\sigma$
$\ell_1$, $\sigma(\ell_1+i-1) = \sigma_2(i)$ for
$\{\ell_1+i-1; i \in D_2\}$.

*Theorem 12.* Let $m \geq n \geq 1$.

*Proof.* It suffices to sh
for every $p \geq 2$. Choose an
result follows from Theorem 5

has the following form:
$s+1, i+s+2, \ldots, i+s+s_2\}$
$1$. Since these $s+1$
ult follows. ■

$n-1)/2\rfloor$. Then the
$, n-1, s, n, s+1, 1$
$s)$. If these $2n$ edges
emaining graph is $\tilde{T}(n,s+1)$.
at the sequence traverses
s: $\{(i,i+s+1) : 1 \le i \le$
s set consists of precisely
ut not $\tilde{T}(n,s+1)$. ■

$\le s \le \lfloor (n-1)/2 \rfloor$. Then
each $s+1$ consecutive

emma 9 when $s = \lfloor (n-1)/2 \rfloor$.
. We then construct an
ing Lemma 10. It remains
vertices in $\sigma$ is dis-
$s\}$ be a set of $s+1$
$S_1 = \{\sigma(j) : j_0 \le j \le$
$\le j \le j_0 + s$, $j$ even$\}$.
secutive integers (cycli-
$j_0$, $S_2$ is a set of con-
or other values of $j_0$, $S_2$
gers with only a single
ing a set of consecutive

If we let $\sigma(j_0) = i$ and
d $S_1 =$
$\sigma(j_0+1) = i + s_3 + 1$



and $s_2 = |S_2|$. Then $s_2 = \lfloor (s+1)/2 \rfloor$, $s \le s_3 \le \lfloor (n-1)/2 \rfloor$, and $S_2$ is a subset of the following set of $s_2 + 1$ consecutive integers $S_2^* = \{i+s_3+1, i+s_3+2, \ldots, i+s_3+s_2+1\}$. Since $s_1 - 1 < s_3 + 1$ and $s_3 + s_2 + 1 < n$, it follows that $S_1 \cap S_2^* = \emptyset$ and thus, the $s+1$ vertices in $S$ are distinct.

Now consider the case when $j_0$ is even. As before let $s_1 = |S_1|$; also let $i = \sigma(j_0+1)$. Then $s_1 = \lfloor (s+1)/2 \rfloor$ and $S_1 = \{i, i+1, i+2, \ldots, i+s_1-1\}$. Now let $\sigma(j_0) = i + s_3$ and $s_2 = |S_2|$. Then $s_2 = \lceil (s+1)/2 \rceil$, $s \le s_3 \le \lfloor (n-1)/2 \rfloor$, and $S_2$ is a subset of the following set of $s_2 + 1$ consecutive integers $S_2^* = \{i+s_3, i+s_3+1, \ldots, i+s_3+s_2\}$. Since $s_1 - 1 < s_3$ and $s_3 + s_2 < n$, it follows that $S_1 \cap S_2^* = \emptyset$ and thus, the $s+1$ consecutive integers in $S$ are distinct. With this observation, the proof is complete. ■

At the risk of belaboring an obvious point, the following sequence determines an Euler circuit of $\tilde{T}(9,2)$ in which every set of 3 consecutive vertices is distinct: 1, 4, 2, 5, 3, 6, 4, 7, 5, 8, 6, 9, 7, 1, 8, 2, 9, 3, 1, 5, 2, 6, 3, 7, 4, 8, 5, 9, 6, 1, 7, 2, 8, 3, 9, 4, 1, 6, 2, 7, 3, 8, 4, 9, 5, 1.

We need one last concept before presenting the proof of our principal theorem. Let $(\sigma_1, D_1)$ be a DP-sequence of length $\ell_1$, and let $(\sigma_2, D_2)$ be a DP-sequence of length $\ell_2$. When $\sigma_1(\ell_1) = \sigma_2(1)$, we define the *splice* of $(\sigma_1, D_1)$ and $(\sigma_2, D_2)$, denoted $(\sigma_1, D_1) \oplus (\sigma_2, D_2)$, as the DP-sequence $(\sigma, D)$ where $\sigma$ has length $\ell = \ell_1 + \ell_2 - 1$, $\sigma(i) = \sigma_1(i)$ for $1 \le i \le \ell_1$, $\sigma(\ell_1 + i - 1) = \sigma_2(i)$ for $1 \le i \le \ell_2$, and $D = D_1 \cup \{\ell_1 + i - 1; i \in D_2\}$.

*Theorem 12.* Let $m \ge n \ge 1$. Then $i(m, n \cdot \infty) \le 1 + i(m,n)$.

*Proof.* It suffices to show that $i(m, n \cdot p) \le 1 + i(m,n)$ for every $p \ge 2$. Choose an arbitrary $p \ge 2$. The desired result follows from Theorem 5 when $m = n$ so we may assume that

$m > n$ .

Now let $t = i(m,n)$ , i.e., $t = \lceil (mn+1)/(m+n) \rceil$ . Then $\lceil (n+1)/2 \rceil \le t \le n$ . If $t \ge n-1$ , the result follows from Theorem 7 since $i(m, n \cdot p) \le i(m, n \cdot \infty) \le n$ . So we may also assume that $t < n-1$ . Then let $s = n - t$ . We observe that $1 \le s \le \lfloor (n-1)/2 \rfloor$ , and in fact $s \ge 2$ .

We now construct a DP-sequence $(\sigma_1, D_1)$ of length $ns + 1$ using the symbols $\{1,2,3,\ldots,n\}$ . The DP-sequence $(\sigma_1, D_1)$ has $n + 1$ distinguished positions $D_1 = \{(i-1)(s) + 1 : 1 \le i \le n+1\}$ . The symbol $i$ occurs in the distinguished position $(i-1)s + 1$ for $i = 1, 2, 3, \ldots, n$ and the symbol $1$ occurs in the distinguished position $ns + 1$ ; note that the symbol $1$ is both the first and last symbol in $\sigma_1$ and that both of these positions are distinguished.

For each $i = 1, 2, 3, \ldots, n$ , and each $j = 1, 2, 3, \ldots, s-1$ , $\sigma_1$ has the symbol $i + j + 1$ in position $(i-1)s + j + 1$ ; the position is not distinguished. We illustrate the definition of $(\sigma_1, D_1)$ when $n = 12$ and $s = 4$ . In this case, $(\sigma_1, D_1)$ is: $\underline{1}, 3, 4, 5, \underline{2}, 4, 5, 6, \underline{3}, 5, 6, 7, \underline{4}, 6, 7, 8, \underline{5}, 7, 8, 9, \underline{6}, 8, 9, 10, \underline{7}, 9, 10, 11, \underline{8}, 10, 11, 12, \underline{9}, 11, 12, 1, \underline{10}, 12, 1, 2, \underline{11}, 1, 2, 3, \underline{12}, 2, 3, 4, \underline{1}$ .

The construction of $(\sigma_2, D_2)$ is simple. We let $\sigma_2$ be a sequence from $\{1,2,3,\ldots,n\}$ which begins and ends with $1$ , determines an Euler circuit of $\underset{\sim}{T}(n,s)$ , and satisfies the requirement that every $s + 1$ consecutive symbols in $\sigma_2$ are distinct. Note that the length of $\sigma_2$ is $n(n-2s) + 1$ . We then let $D_2 = \{1,2,3,4,\ldots,n(n-2s)+1\}$ , i.e., every position in $(\sigma_2, D_2)$ is distinguished.

Now let $(\sigma, D)$ be the splice $(\sigma_1, D) + (\sigma_2, D_2)$ . Note that the length of $\sigma$ is $ns + n(n-2s) + 1 = n(n-s) + 1 = nt + 1$ . Also note that $D = \{(i-1)s + 1 : 1 \le i \le n+1\} \cup \{i : n(n-2s) + 1 \le i \le nt+1\}$ .



The next step in the argum
$(\sigma,D)$ with the criteria given
$(\sigma,D)$ produces an interval rep
To see that this statement hol
$(j_1,j_2)$ from $\{1,2,3,\ldots,n\}$ .
then we may set $\beta = j_2$ and ob
$j_1 \in \{\sigma(k) : k_\beta \le k \le k_{\beta+1}\}$ .
$\{j_2 - j : 0 \le j \le s-1\}$ then we ma
and $\beta = n$ when $j_1 = 1$ and
$k_{\beta+1}\}$ and $j_1 = \sigma(k_{\beta+1})$ . If
then $(j_2,j_1)$ is an edge in
integer $j$ with $1 \le j \le n(n-$
$\sigma_2(j+1) = j_1$ . We may then s
$\sigma(k_\beta) = j_2$ and $\sigma(k_{\beta+1}) = j_1$

This completes the proof
mines an interval representati

Furthermore, we observe t
except $1$ , is used exactly
$1$ is used $t+1$ times in $\sigma$
vals corresponding to vertices
tion of $(\sigma,D)$ in order to ob
$K(m,n \cdot p)$ . For each vertex
vals. One of these intervals,
overlap intervals for $s+1$
these $s+1$ vertices will be
the other $t-1$ intervals wh
$I(a))$ will overlap exactly on
interval and the vertex to whi
each $i$ .

For $j = 1, 2, \ldots, m$ w
overlaps the intervals corresp
$\sigma(js+1)$ for each of the leve

The next step in the argument is to compare the DP-sequence $(\sigma,D)$ with the criteria given in Theorem 8 and observe that $(\sigma,D)$ produces an interval representation of $K(n,n\cdot(p-1))$. To see that this statement holds, consider an ordered pair $(j_1,j_2)$ from $\{1,2,3,\ldots,n\}$. If $j_1 \in \{j_2+j : 0 \le j \le s\}$, then we may set $\beta = j_2$ and observe that $j_2 = \sigma(k_\beta)$ and $j_1 \in \{\sigma(k) : k_\beta \le k \le k_{\beta+1}\}$. Similarly, if $j_1 \in \{j_2 - j : 0 \le j \le s-1\}$ then we may set $\beta = j_1 - 1$ when $j_1 > 1$ and $\beta = n$ when $j_1 = 1$ and observe that $j_2 \in \{\sigma(k) : k_\beta \le k \le k_{\beta+1}\}$ and $j_1 = \sigma(k_{\beta+1})$. If neither of these conditions hold, then $(j_2,j_1)$ is an edge in $\underset{\sim}{T}(n,s)$ and there exists an integer $j$ with $1 \le j \le n(n-2s)$ so that $\sigma_2(j) = j_2$ and $\sigma_2(j+1) = j_1$. We may then set $\beta = n + j$ and observe that $\sigma(k_\beta) = j_2$ and $\sigma(k_{\beta+1}) = j_1 \in \{\sigma(k) : k_\beta \le k \le k_{\beta+1}\}$.

This completes the proof of our claim that $(\sigma,D)$ determines an interval representation of $K(n,n\cdot(p-1))$.

Furthermore, we observe that each symbol in $\{1,2,3,\ldots,n\}$, except $1$, is used exactly $t$ times in $\sigma$, and the symbol $1$ is used $t+1$ times in $\sigma$. We now show how to add intervals corresponding to vertices in $A$ to an interval representation of $(\sigma,D)$ in order to obtain an interval representation of $K(m,n\cdot p)$. For each vertex $a \in A$, we will assign $t$ intervals. One of these intervals, which we denote $I(a)$, will overlap intervals for $s+1$ distinct vertices from each $B_i$; these $s+1$ vertices will be the same for each $i$. Each of the other $t-1$ intervals which correspond to $a$ (other than $I(a)$) will overlap exactly one interval from each $B_i$; this interval and the vertex to which it corresponds are the same for each $i$.

For $j = 1, 2, \ldots, m$ we choose an interval $I(a_j)$ which overlaps the intervals corresponding to $\sigma((j-1)s+1)$ and $\sigma(js+1)$ for each of the levels in the representation. Note that $I(a_j)$ overlaps a set of $s+1$ intervals corresponding to $s+1$ distinct vertices in $B_i$ for each $i$.

For each $j = 1, 2, \ldots, m$, we then choose $t-1$ "points" which overlap intervals corresponding to the $n-(s+1) = t-1$ vertices in $B_i$ not already overlapped by $I(a_j)$. This assignment is easily accomplished since the first $n$ distinguished positions in $(\sigma, D)$ contain $\{1,2,3,\ldots,n\}$. With this observation, the proof is complete. ∎

We illustrate the preceding theorem for $m = 7$, $n = 5$, and $p = 2$. For clarity, the points corresponding to vertices in $A$ are omitted.

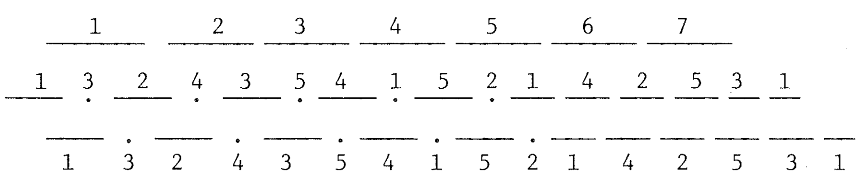



Figure 10.

For emphasis, we also state as a formal theorem the following alternate form of Theorem 12.

*Theorem 13.* If $p, n_1, n_2, n_3, \ldots, n_p$ are integers with $p \geq 2$ and $n_1 \geq n_2 \geq \ldots \geq n_p$, then

$$i(n_1, n_2, n_3, \ldots, n_p) \leq 1 + i(n_1, n_2).$$

4. Concluding Remarks.

It should be noted that the inequality in Theorems 12 and 13 is best possible as the following result due to D. West [5] implies.



*Theorem 14* [5]. If $n \geq 3$ anc
$i(m, n\cdot\infty) = 1 + i(m,n)$.

The construction used in
the determination of $i(m,n\cdot\infty$
of DP-sequences. We announce
solved completely the problem
values of $m$ and $n$. The pr
appear elsewhere.

*Theorem 15.* Let $m, n$ be int

$$i(m, n\cdot\infty) = \begin{cases} 1 + i(m, \\ 1 + i(m, \\ i(m,n) \end{cases}$$

*REFERENCES*

1. J.R. Griggs and D.B. West,
number of a graph, I, *SIAM*
to appear.
2. J.R. Griggs, Extremal valu
graph, II, *Discrete Mathem*
3. M. Matthews and William T.
Complete Multipartite grap
of SE SIAM, April, 1978.
4. William T. Trotter, Jr. ar
Multiple Interval Graphs,
(1979), 205-211.
5. D. West, Personal Communic

University of South Carolina

*Theorem 14* [5]. If $n \geq 3$ and $m = n^2 - n - 1$, then $i(m, n \cdot \infty) = 1 + i(m,n)$.

The construction used in Theorem 12 suggests strongly that the determination of $i(m, n \cdot \infty)$ for $m \geq n$ rests on properties of DP-sequences. We announce that the authors and D. West have solved completely the problem of determining $i(m, n \cdot \infty)$ for all values of $m$ and $n$. The proof of the following result will appear elsewhere.

*Theorem 15.* Let $m, n$ be integers with $m \geq n$. Then

$$i(m, n \cdot \infty) = \begin{cases} 1 + i(m,n) & \text{if } n \geq 3 \text{ and } m = n^2 - n - 1 \\ 1 + i(m,n) & \text{if } m = 7 \text{ and } n = 5 \\ i(m,n) & \text{otherwise.} \end{cases}$$

*REFERENCES*


1. J.R. Griggs and D.B. West, Extremal values of the interval number of a graph, I, *SIAM J. Algebraic and Discrete Methods*, to appear.
2. J.R. Griggs, Extremal values of the interval number of a graph, II, *Discrete Mathematics*, 28(1979), 37-47.
3. M. Matthews and William T. Trotter, Interval Number of the Complete Multipartite graph, Presented at 2nd Annual Meeting of SE SIAM, April, 1978.
4. William T. Trotter, Jr. and Frank Harary, On Double and Multiple Interval Graphs, *Journal of Graph Theory*, 3 (1979), 205-211.
5. D. West, Personal Communication.


University of South Carolina