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26

UNIT DISTANCES IN THE EUCLIDEAN PLANE

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ABSTRACT Consider a set of n points in the Euclidean plane and let $U(n)$ denote the maximum number of pairs of points which can be at unit distance. Erdös observed that $U(n) > n^{1+c/(\log \log n)}$ and conjectured that $U(n)$ has an upper bound of the same form. Progress on this conjecture has been slow, and to date the best known upper bound, $U(n) < n^{1.499}$, is due to Beck and Spencer. In this paper, we modify techniques first used to solve extremal problems involving configurations of points and lines in the plane to obtain the improved bound: $U(n) < cn^{4/3}$.

1. Introduction

Combinatorial problems in discrete geometry have been investigated extensively by numerous researchers. The reader is encouraged to consult Erdös's survey papers **[2–4]** and Moser's annual summary **[6]** for a compilation of results and an extensive bibliography. Here, we concentrate on one of the oldest and most tantalizing problems in this area: What is the maximum number $U(n)$ of unit distances determined by n points in the Euclidean plane!

Consideration of the lattice points shows that $U(n) > n^{1+c/(\log \log n)}$ and Erdös conjectures that this inequality is essentially best possible. However, progress on upper bounds has been slow. In **[5]**, Józsa and Szemerédi showed that $U(n) = o(n^{3/2})$. This result was improved by Beck and Spencer **[1]**, who showed that $U(n) < n^{1.499}$. In this paper, we will show that $U(n) < cn^{4/3}$. In order to establish this result, we will employ techniques similar to those developed by Szemerédi and Trotter in **[7]** and **[8]**. The results in these papers provide inequalities involving incidences between points and lines. We will need a number of modifications in order to obtain results for points and circles.

2. The Covering Lemma

Assume that a pair of perpendicular lines has been chosen and used to establish a coordinatization of the plane. The following lemma is proved

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in [7]:

LEMMA 1 *Let r_1 and r_2 satisfy $r_2 \geq 256 r_1$ and let P be a set of n points in the plane. Then there exists a family Q of squares so that:*

(i) *The sides of the squares in Q are parallel to the coordinate axes.*
(ii) *Each square in Q contains at least r_1 but no more than r_2 points from P.*
(iii) *No point in the plane belongs to the interior of two or more squares from Q.*
(iv) *At least $n/16$ of the points in P are covered by the squares in Q.* □

In this paper, we will require a slight modification of this covering lemma. Instead of covering points with squares, we will use rectangles for which the ratio between the width and the height is some fixed constant. It is easy to see that the lemma remains valid with this modification since a linear transformation can be used to interchange squares with rectangles of the desired shape.

3. The principal theorem

In this section, we prove the existence of an absolute constant c for which $U(n) < cn^{4/3}$. The result will follow as an easy corollary to a somewhat more technical result. Consider two squares resting on the x-axis. Each side has length 10^{-6} and the centers of the squares are exactly 1 unit apart. We then denote by $U(n, t)$ the maximum number of unit distances which can occur between a set X of n points in one square and a set Y of t points in the other. For convenience, we consider the points in X as belonging to the leftmost square.

The approximation of $U(n, t)$ is relatively simple if one of the parameters is quite large in comparison with the other since the optimal configuration (up to a constant factor) places all the points in one set on a unit circle determined by one point in the other set. However, when $\sqrt{n} \leq t \leq n^2$, the problem is much more difficult. Here, we prove the following inequality:

THEOREM 1 *There exists an absolute constant c so that $U(n, t) < cn^{2/3}t^{2/3}$ whenever $\sqrt{n} \leq t \leq n^2$.*

PROOF Let $M_0 = 1$. Then define $M_{i+1} = 10M_i^{10}$ for each $i \geq 0$. We show that this inequality is valid when $c = M_{10^{12}}$.

The argument is by contradiction. We suppose the result is false and choose a counterexample with $n + t$ as small as possible. For this configuration, $U(n, t) \geq cn^{2/3}t^{2/3}$. The remaining part of the proof is subdivided into several parts. First, we introduce some appropriate labels. Second, we present several counting lemmas. Third, we describe in a series of steps a method for producing a subconfiguration satisfying several key properties. Finally, we count the number of times arcs cross and show that it is more than the total number of pairs of arcs.

Label the points in X as $x_1, x_2, \ldots, x_n$ and the points in Y as $y_1, y_2, \ldots, y_t$. For each $i = 1, 2, \ldots, n$, we let d_i count the number of unit distances from x_i to Y. Similarly, for each $j = 1, 2, \ldots, t$, we let e_j count the number of unit distances from y_j to X. For simplicity, we write $U = U(n, t)$. Note that

$$\sum_{i=1}^{n} d_i = \sum_{j=1}^{t} e_j = U.$$

Now a pair of points in one of our two sets can belong to at most one unit circle whose center is a point in the other set. Thus,

$$\sum_{i=1}^{n} \binom{d_i}{2} \leq \binom{t}{2}.$$

We conclude that

$$t^2 \geq \frac{1}{10} \sum_{i=1}^{n} d_i^2 \geq \frac{1}{10n} \left(\sum_{i=1}^{n} d_i \right)^2 \geq \left(\frac{c^2}{10} \right) n^{1/3} t^{4/3}.$$

From this, it follows that (being generous) $c^2\sqrt{n} < t$. Since the argument is dual, we know that $c^2\sqrt{t} < n$, i.e. $t < n^2/c^4$. The important observation to make here is that since c is extremely large, we can apply the inductive hypothesis to subsets of size n/M_i and t/M_j respectively, where M_i and M_j are large – but relatively small in comparison to c; for example, when $i, j \leq 1000$. That is, we can be assured that $\sqrt{(n/M_i)} \leq t/M_i \leq (n/M_i)^2$ and that the number of unit distances determined by these subsets is less than $c(n/M_i)^{2/3}(t/M_j)^{2/3}$.

Another easy conclusion that can be drawn from the inequality

$$\binom{t}{2} \geq \sum_{i=1}^{n} \binom{d_i}{2}$$

is that $U(n, t) \geq 10t^2$. Dually, $U(n, t) \geq 10n^2$. Note that these inequalities hold for all n, t with $\sqrt{n} \leq t \leq n^2$. Furthermore, we observe that the function $U(n, t)$ increases monotonically in both coordinates.

Next, we present several claims concerning unit distances between subsets of X and Y. In each case, the proof is an elementary counting argument. Furthermore, we make no attempt to obtain the best possible inequalities. Instead, we present the claims in simple form to facilitate their use later in our proof. Let d denote the average value of d_i, i.e.

$$d=\frac{1}{n}\sum_{i=1}^{n} d_i.$$

CLAIM 1 Let $i\leqslant 100$ and let A and B be subsets of X and Y respectively. Suppose there are at least U/M_i unit distances in $A\cup B$. Then there are at least n/M_{i+1} points in A each having at least d/M_{i+1} unit distances in B.

PROOF Let A_0 denote the subset of A consisting of those points with at least d/M_{i+1} unit distances in B. Suppose that $|A_0|<n/M_{i+1}$. Then the number of unit distances in $A_0\cup B$ is less than $c(n/M_{i+1})^{2/3}t^{2/3}<U/2M_i$. This inequality follows from the observation that $M_{i+1}=10M_i^{10}>2^{3/2}M_i^{3/2}$. However, the number of unit distances in $(A-A_0)\cup B$ is less than $nd/M_{i+1}<U/2M_i$. These inequalities imply that there are less than U/M_i unit distances in $A\cup B$. The contradiction completes the proof. □

CLAIM 2 Let $i\leqslant 100$. Suppose that A and B are subsets of X and Y respectively with $|A|\geqslant n/M_i$. Suppose further that each point in A has at least d/M_i unit distances in B. Then let $K=16M_i^2$ and let $B=B_1\cup B_2\cup\ldots\cup B_K$ be an arbitrary partition into subsets of equal size. Then there exists a pair (α,β) with $1\leqslant\alpha<\beta\leqslant K$ and a subset $A(\alpha,\beta)$ of A with $|A(\alpha,\beta)|\geqslant N/M_{i+1}$ so that each point in $A(\alpha,\beta)$ has at least d/M_{i+1} unit distances in B_α and at least d/M_{i+1} unit distances in B_β.

PROOF For each $\alpha=1,2,\ldots,K$, let A_α denote those points in A which have at least d/M_{i+1} unit distances in B_β if and only if $\alpha=\beta$. Then each point in A_α has at least $d/M_i-(K-1)d/M_{i+1}>d/2M_i$ unit distances in B_α.

Now suppose there is some $\alpha\leqslant K$ with $|A_\alpha|=xn$ and $x\geqslant 8M_i/K^2$. Then we may conclude that $xnd/2M_i=|A_\alpha|\,d/2M_i<c\,|A_\alpha|^{2/3}(t/K)^{2/3}\leqslant Ux^{2/3}/K^{2/3}=ndx^{2/3}/K^{2/3}$, and thus $x<8M_i/K^2$. The contradiction shows that $x<8M_i/K^2$ for all α. Therefore, $A_0=A_1\cup A_2\cup\ldots\cup A_k$ contains at most $K(8M_i/K^2)n$ points. However, $K(8M_i/K^2)n=n/2M_i$. Thus $|A-A_0|\geqslant n/2M_i$.

Since

$$(n/2M_i)\Big/\binom{K}{2}>n/M_{i+1},$$

the existence of the desired set $A(\alpha,\beta)$ follows by the box principle and

the observation that, for each point a from A, there must be some B_α so that a has at least d/M_{i+1} unit distances in B_α. □

CLAIM 3 Let $i\leqslant 100$. Suppose that A and B are subsets of X and Y respectively with $|A|\geqslant n/M_i$. Suppose further that each point in A has at least d/M_i unit distances in B. Then let K be an arbitrary positive integer and let $A_1, A_2, A_3,\ldots, A_K$ be disjoint subsets of A whose union contain at least half the points in A. If no point in B has unit distances in two or more of the A_αs, then there is at least one α for which $|A_\alpha|\geqslant n/M_{i+2}$.

PROOF Suppose to the contrary that $|A_\alpha|<n/M_{i+2}$ for every α. Let $A_0=A_1\cup A_2\cup\ldots\cup A_K$. Then we can form new partitions of A_0 by merging subcollections of the A_αs. It is easy to see that there exists a partition of this type $A_0=A_1'\cup\ldots\cup A_{K'}'$, where $n/2M_{i+1}\leqslant|A_\alpha'|\leqslant 2n/M_{i+1}$ for each α. Then for each $\alpha=1,2,\ldots,K'$, let B_α denote the subset of B with at least one unit distance in A_α'. Let B_α' denote those points in B_α with two or more unit distances in A_α'.

The number of unit distances in $A_\alpha'\cup B_\alpha$ is at least $nd/2M_iM_{i+1}$ and the number of unit distances in $A_\alpha'\cup(B_\alpha-B_\alpha')$ is at most $|B_\alpha-B_\alpha'|\leqslant t<nd/4M_iM_{i+1}$. On the other hand, the number of unit distances in $A_\alpha'\cup B_\alpha'$ is less than $c(2n/M_{i+1})^{2/3}\,|B_\alpha'|^{2/3}$. Thus, we must have

$$\frac{nd}{4M_iM_{i+1}}<c\left(\frac{2n}{M_{i+1}}\right)^{2/3}|B_\alpha'|^{2/3}.$$

This requires $|B_\alpha'|>t/M_{i+1}^{1/2}$. Since this inequality holds for every $\alpha=1,2,\ldots,K'$, we conclude that $K'<M_{i+1}^{1/2}$. However, since each A_α' contains at most $2n/M_{i+1}$ points and their union contains at least $n/2M_i$ points, we conclude that $K'\geqslant M_{i+1}/4M_i>M_{i+1}^{1/2}$. The contradiction completes the proof. □

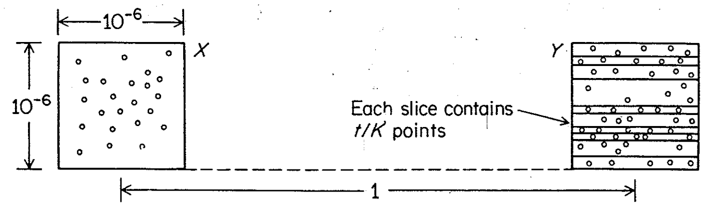
The reader should note that these three claims have dual forms in which the roles played by the two squares and the sets X and Y are reversed. Now we are ready to use these claims to obtain a subconfiguration satisfying several essential properties.

In these arguments, we will be subdividing rectangles by horizontal or vertical lines. When the subdivision is by horizontal lines we call the pieces *strips* and use the term *slices* when the subdivision is by vertical lines. In each case the horizontal dimension of a rectangle is called its *width* and the vertical dimension its *height*.

Step 1 Apply Claim 1 and choose a subset X_1 of X so that $|X_1|\geqslant n/M_1$ and each point in X_1 has at least d/M_1 unit distances in Y.

Step 2 Apply Claim 2 with $i=1$. Divide Y into K subsets of size t/K

by dividing the right most square into K strips. The height of the strips may vary, but each is to contain t/K points from Y.



We obtain two strips on the right each having at most t points and a subset X_2 of X containing at least n/M_2 points each having at least n/M_2 unit distances in each of these two strips.

Step 3 Apply Claim 2 again to each of the two strips obtained in Step 2. We obtain four strips each of size at most t and a subset X_4 of X containing at least n/M_4 points each having at least d/M_4 unit distances in each of the four strips.

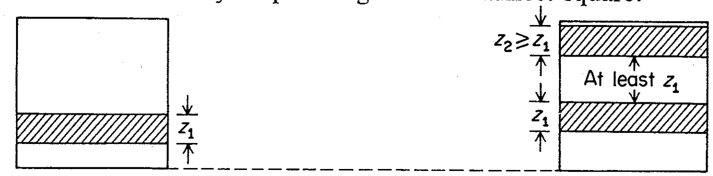
Step 4 Label the four strips from bottom to top as S_1, S_2, S_3, S_4 and let s_1, s_2, s_3, s_4 denote their respective heights. Without loss of generality, we assume that $s_2 \leq s_3$. Otherwise, turn the entire plane upside down. Delete from consideration all points in Y except those in strips S_2 and S_4. Note that these two strips each contain at least t/M_5 of the points in Y, and that the gap between them is at least s_2.

Step 5 Try to find a strip of thickness s_2 which covers at least n/M_{10} of the points in X_4. If this can be done, go to Step 8.

Step 6 In this case, we are unable to cover n/M_{10} of the points in X_4 with a strip of height s_2. Now consider the n/M_4 points in X_4 and the set of at least t/M_4 points in S_2. In these two sets there are at least $nd/M_4M_4 > U/M_5$ unit distances. So it follows that there is a subset Y_6 of S_2 with $|Y_6| \geq t/M_6$ so that each point in Y_6 has at least e/M_6 unit distances in X_4.

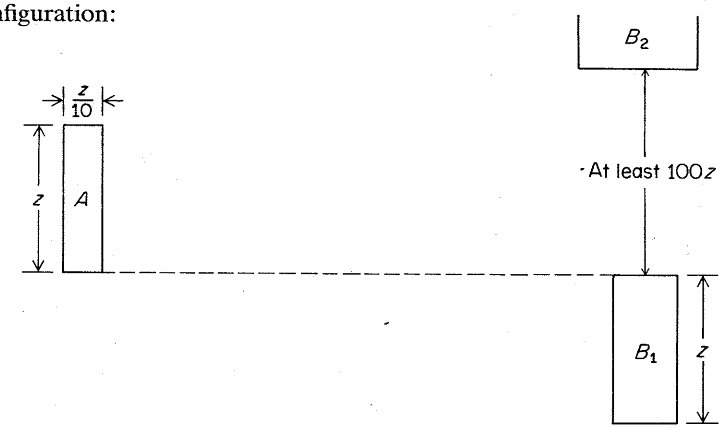
Step 7 Apply Claim 2 to obtain four strips in the left most rectangle each containing at least n/M_{10} points from X and a subset Y_{10} of S_2 of size at least t/M_{10} so that each point in Y_{10} has at least e/M_{10} unit distances in each of these four strips. Label these strips from bottom to top as R_1, R_2, R_3, R_4 and let r_1, r_2, r_3 and r_4 denote their respective heights. Again, we may assume that $r_2 \leq r_3$.

Step 8 From either Step 5 or Step 7 we have a strip of height z_1 in one square and two strips in the other square of heights z_1 and z_2 respectively with the two strips separated by a gap of at least z_1. Furthermore, each of these three strips contains at least an M_{10}th of the points in the entire square. The points in the solitary strip have at least d/M_{10} (e/M_{10} if the solitary strip is in the rightmost square) unit distances in each of the two strips in the other square. For convenience we may assume that the solitary strip belongs to the leftmost square.



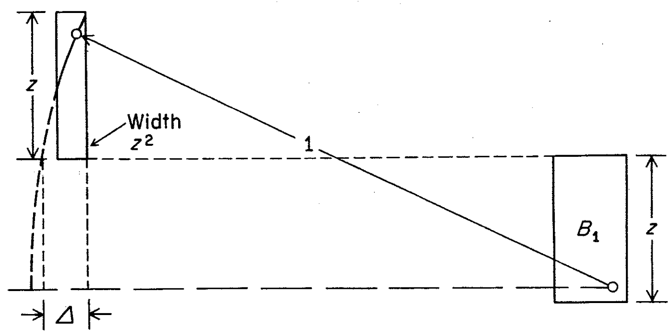
Next we partition the solitary strip vertically into slices whose width is $z_1/10$. Label the slices from left to right as $A_1, A_2, \ldots, A_K$. Since no point in either of the strips in the right square has unit distances in two or more of these subrectangles unless they are consecutive, we may apply Claim 3 to the larger of $A_1 \cup A_3 \cup A_5 \cup \ldots$ and $A_2 \cup A_4 \cup A_6 \cup \ldots$.

Step 9 So now we have a set A of size at least n/M_{12} so that all points in A are contained in a rectangle whose height is z_1 and whose width is $z_1/10$. Every point in A has at least d/M_{12} unit distances in each of two sets B_1 and B_2 which are contained respectively in strips of thickness z_1 and z_2. Furthermore, these strips are separated by a gap of at least z_1. It is clear that all points in B_1 which have a unit distance in A can be covered with a rectangle of height z_1 and width $3z_1/10$. Since we do not know the height of the rectangle containing B_2 and it could be much larger than z_1, we make no such claim for B_2. Now choose a strip S of the rectangle containing A of height $z_1/1000 = z$ so that S contains at least n/M_{13} of the points in A. Then apply Claim 1 to choose a subset B_1' of B_1 of size t/M_{15} with each point in B_1' having at least e/M_{15} unit distances in S. By the box principle, we can then choose a strip in B_1 containing at least t/M_{16} points each having at least e/M_{15} unit distances in S with the height of the strip in B_1 also at most $z_1/1000$. Applying Claims 1 and 3 again and rotating the configuration if necessary, we have the following configuration:



Each point in A has at least d/M_{18} unit distances in B_1 and B_2.

Next, we divide A vertically into slices of width z^2. Note that no point of B_1 can be adjacent to points from non-consecutive slices:



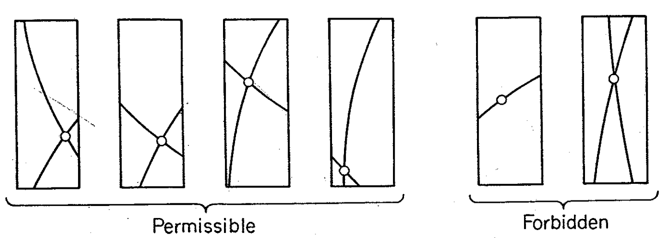
Note also that $\Delta < 2z^2$.

By Claim 3, we conclude after relabeling that we have a set A in a rectangle whose height is z and whose width is z^2 with $|A| \geqslant n/M_{20}$. Furthermore, we have two strips B_1 and B_2 with the thickness of B_1 also z and the gap between B_1 and B_2 at least $100z$. Also, each point in A has at least d/M_{20} unit distances in B_1 and in B_2.

Next, we present the essential idea for counting crossings between arcs of the unit circles determined by points in this subconfiguration. Let A' be a subrectangle of A. We say A' is *similar* to A if both rectangles have the same ratio between their height and their width and if the height of A' is at most half the height of A.

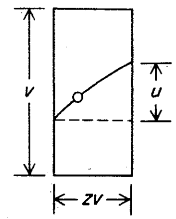
THREE SIDES LEMMA *Let A' be a subrectangle of A with A' similar to A. Then let p be any of the points in A' and consider any two unit circles determined by a point in B_1 and a point in B_2. Then these circles intersect at least three of the sides of A'. Furthermore, the circle determined by the point in B_1 cannot intersect both vertical sides of A'.*

We illustrate the statement of this lemma with the following diagram:



PROOF To prove the lemma, we proceed as follows. We can easily

dismiss the first of the two forbidden configurations. Let u and v be as shown below:



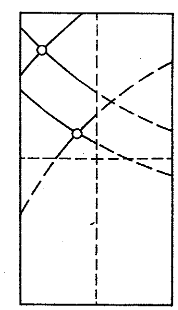
Since the arc hits the vertical sides, we must have $zv < 2u^2 \leqslant 2v^2$. But this implies that $z < 2v$, which is a contradiction.

For the second configuration, we consider the angle between the tangent lines to the arcs at the point p. For this configuration to arise, it is clear that this angle is less than $10z$. On the other hand, since B_1 has thickness z and the gap between B_1 and B_2 is at least $100z$, we know the angle between these tangent lines is at least $50z$. Thus, neither forbidden configuration can occur. □

The next lemma is quite elementary but involves the systematic examination of several cases. For the sake of brevity, we provide only a sketch of the argument and leave the details to the reader.

INTERSECTION LEMMA *Let A' be similar to A. Divide A' into four similar rectangles by perpendicular bisectors to the sides. Then let p_1 and p_2 be points in the same quarter of A'. For each of these points, choose a pair of unit circles passing through them as determined by points in B_1 and B_2. Then at least one of the arcs through p_1 crosses one or both of the arcs through p_2 at a point inside A'.*

PROOF The arcs through p_1 partition the quarter into four regions. The result is immediate unless the region containing p_2 touches three of the four sides of the quarter rectangle. A dual statement holds for the regions determined by P_2. For example,



By analyzing the four quadrants individually, it is easy to see the desired crossing exists. □

We are now ready to make the final computation. We apply the Covering Lemma in Section 2 to the points in A with rectangles similar to A. We use $r_1 = d/M_{23}$ and $r_2 = d/M_{22}$.

Now consider a rectangle R used in the covering. Choose a quarter rectangle R' of R which contains at least $r_1/4 \geq d/M_{24}$ points. Each of these points has at least d/M_{20} unit distances in B_1 and at least d/M_{20} unit distances in B_2. So for each point p in R, there are at least d/M_{21} points in B_1 for which p is the unique point of A in R at unit distance. A similar statement holds for B_2. There are at least d^2/M_{24} pairs of points from A which belong to R'. For each such pair there are at least d^2/M_{22} positions where arcs cross inside R. These crossings are wasted since they occur where there is no point in A. So inside each rectangle R, there are at least d^4/M_{25} wasted crossings. However, there are at least

$$\frac{n}{16M_{20}} \frac{d}{M_{22}} > n/d$$

rectangles in the covering. Thus there are at least

$$nd^3/M_{25} > \frac{c^3t^2}{M_{25}} > \binom{t}{2}$$

wasted crossings altogether. Clearly this is impossible. The contradiction completes the proof of Theorem 1. □

It is easy to see that the following results may be obtained as easy corollaries from Theorem 1 using nothing more than the box principle.

COROLLARY 1 *There exists an absolute constant c so that whenever $\sqrt{n} \leq t \leq n^2$ and X and Y are sets of size n and t respectively, the number of pairs $(x, y) \in X \times Y$ which are at unit distance is less than $cn^{2/3}t^{2/3}$.* □

COROLLARY 2 *There exists an absolute constant c so that $U(n)$, the maximum number of unit distances determined by n points in the plane, satisfies the inequality $U(n) < cn^{4/3}$.* □

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References

1. J. Beck and J. Spencer, Unit distances. To appear.
2. P. Erdös, On some problems of elementary and combinatorial geometry. *Annali Mat., Ser. 4* **103** (1974), 99–108.
3. P. Erdös, Some applications of graph theory and combinatorial methods to number theory and geometry. In *Algebraic Methods in Graph Theory*, Coll. Math. Soc. T. Bolyai no. 25, Elsevier, Amsterdam, 1981, pp. 131–148.
4. P. Erdös, Problems and results in combinatorial geometry. *Proc. N.Y. Acad. Sci.* to appear.
5. S. Józsa and E. Szemerédi, The number of unit distances in the plane. In *Infinite and Finite Sets* (A. Hajnal *et al.*, eds), North Holland, Amsterdam, 1975.
6. W. Moser, Research problems in discrete geometry. Mimeograph notes, 1981.
7. E. Szemerédi and W. Trotter, Jr, Extremal problems in discrete geometry. *Combinatorica* **3** (1983), 381–397.
8. E. Szemerédi and W. Trotter, Jr, A combinatorial distinction between the Euclidean and projective plane. *Europ. J. Combinat.* to appear.