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THE DIMENSION OF THE CARTESIAN PRODUCT OF PARTIAL ORDERS*

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If P and Q are partial orders, then the dimension of the cartesian product $P\times Q$ does not exceed the sum of the dimensions of P and Q. There are several known sufficient conditions for this bound to be attained. On the other hand, the only known lower bound for the dimension of a cartesian product is the trivial inequality $\dim(P\times Q)\geqslant\max\{\dim P, \dim Q\}$. In particular, if P has dimension n, we know only that $n\leqslant\dim(P\times P)\leqslant 2n$. In this paper, we show that for each $n\geqslant 3$, the crown S_n^0 is an n-dimensional partial order for which $\dim(S_n^0\times S_n^0)=2n-2$. No example for which $\dim(P\times Q)<\dim P+\dim Q-2$ is known.

On sait que la dimension du produit cartésien $P\times Q$ d'ordres partiels P et Q ne dépasse pas la somme des dimensions de P et de Q. On connaît plusieurs conditions suffisantes pour que l'égalité ait lieu. D'un autre côté, le seul minorant connu de $\dim(P\times Q)$ est $\max(\dim P, \dim Q)$. Dans le cas particulier où P est de dimension n, on ne connaît que les égalités $n\leqslant\dim(P\times Q)\leqslant 2n$. Nous montrons dans cet article que, pour tout $n\geqslant 3$, la couronne S_n^0 est un ordre partiel de dimension n pour lequel $\dim(S_n^0\times S_n^0)=2n-2$. On ne connaît aucun exemple pour lequel $\dim(P\times Q)<\dim P+\dim Q-2$.

1. Introduction

In the past several years, a number of researchers have investigated the dimension of partial orders. We refer the reader to [5] for an extensive bibliography and a concise summary of the known results in this area. Here, we provide only the basic definitions necessary to discuss the dimension of cartesian products. The *dimension* of a partial order P is the least integer t for which P is the intersection of t linear orders [3]. An incomparable pair (x, y) in P is called a *nonforced* pair if $z<y$ whenever $z<x$ and $x<w$ whenever $y<w$. A linear extension L *reverses* an incomparable pair (x, y) when $y<x$ in L. Then the dimension of P is the least positive integer t for which there exist t linear extensions of P so that each nonforced pair is reversed in at least one of the extensions [5].

The *cartesian product* of two partial orders P and Q, denoted by $P\times Q$, is the set of pairs (p, q) with $p\in P$ and $q\in Q$ ordered by $(p_1, q_1)\leqslant(p_2, q_2)$ if and only if $p_1\leqslant p_2$ in P and $q_1\leqslant q_2$ in Q. The dimension of a partial order P may then be

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alternately defined [7] as the least t for which P can be embedded in the cartesian product of t chains. Consequently, we have the following elementary inequality.

Fact 1. $\dim(P \times Q) \leqslant \dim(P) + \dim(Q)$ *for every* P, Q.

There are many instances in which this bound is achieved. Here is one such condition due to Baker [1].

Theorem 1. *If P and Q have distinct bounds, then* $\dim(P \times Q) = \dim(P) + \dim(Q)$.

Investigations into the dimension of cartesian products, especially when $\dim(P \times Q) = \dim(P) + \dim(Q)$, have already produced substantial results. Kelly [4] devised a novel construction, called a dimension product, which could be used to explicitly determine an irreducible partial order contained in $P \times Q$ and having the same dimension as $P \times Q$. Trotter and Ross [13, 14] gave a general method for constructing irreducible partial orders with prescribed parameters starting with Kelly's dimension product.

However, in this paper our primary concern will be with lower bounds for $P \times Q$. Since P and Q are suborders of $P \times Q$, we have the following trivial bound.

Fact 2. $\max\{\dim(P), \dim(Q)\} \leqslant \dim(P \times Q)$ *for all* P, Q.

We conjecture that this bound is best possible.

Conjecture 1. For every m, n with $1 \leqslant m \leqslant n$, there exist partial orders P, Q with $\dim(P) = m$, $\dim(Q) = n$, and $\dim(P \times Q) = n$.

In addition, we believe the following special case is also valid.

Conjecture 2. For every $n \geqslant 2$, there exists a partial order P with $\dim(P) = \dim(P \times P) = n$.

Little progress has been made on these conjectures in the past several years. Here, we will show that for each n, there is an n-dimensional partial order P for which $\dim(P \times P) = 2n - 2$. This result was announced in [10]. We will also discuss some potential approaches to the general problem.

2. The cartesian product of crowns

For each $n \leqslant 3$, the crown S_n^0 is the poset with n maximal elements $\{a_1, a_2, \ldots, a_n\}$, n minimal elements $\{b_1, b_2, \ldots, b_n\}$, and ordering $b_i < a_j$ if and only if $i \neq j$. This partial order can be viewed as the 1-element and $n-1$-element subsets of an n-element set ordered by inclusion. In dimension theory, S_n^0 is known as the 'standard' example of a n-dimensional partial order. The fact that S_n^0 is n-dimensional follows immediately from the observations that S_n^0 has n nonforced pairs $\{(b_i, a_i): 1 \leq i \leq n\}$, but that any linear extension of S_n^0 reverses at most one of these nonforced pairs. We refer the reader to [2, 8, 9] for discussions of more general classes of partial orders containing S_n^0 as a special case.

Before proceeding to the determination of the dimension of $S_n^0 \times S_n^0$, we pause to make a few simplifying remarks. Consider the suborder P of $S_n^0 \times S_n^0$ consisting of the n^2 maximal elements $\{(a_i, a_j): 1 \leq i,\ j \leq n\}$ and the n^2 minimal elements $\{(b_i, b_j): 1 \leq i, j \leq n\}$. We claim that $\dim(P) = \dim(S_n^0 \times S_n^0)$. This fact follows immediately from the observation that all the nonforced pairs of $S_n^0 \times S_n^0$ are pairs of the form $((b_i, b_j), (a_k, a_l))$ where either $i = k$ or $j = l$. We then let N_P denote the nonforced pairs in P.

We find it convenient to visualize the elements of P as being arranged in rows and columns with the maximal elements on one plane and the minimal elements on another. In this way, each minimal element (b_i, b_j) is incomparable with all maximal elements appearing in the ith row and jth column. See Fig. 1.

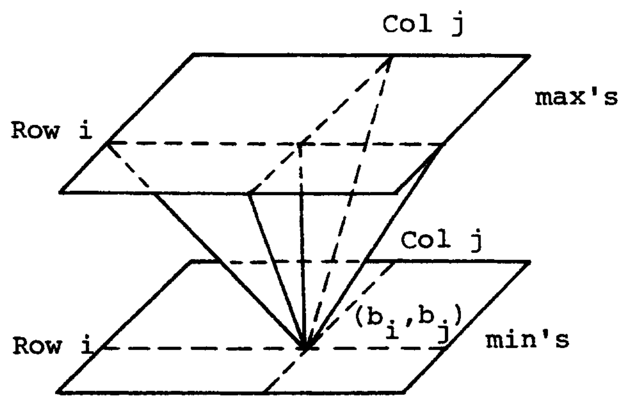
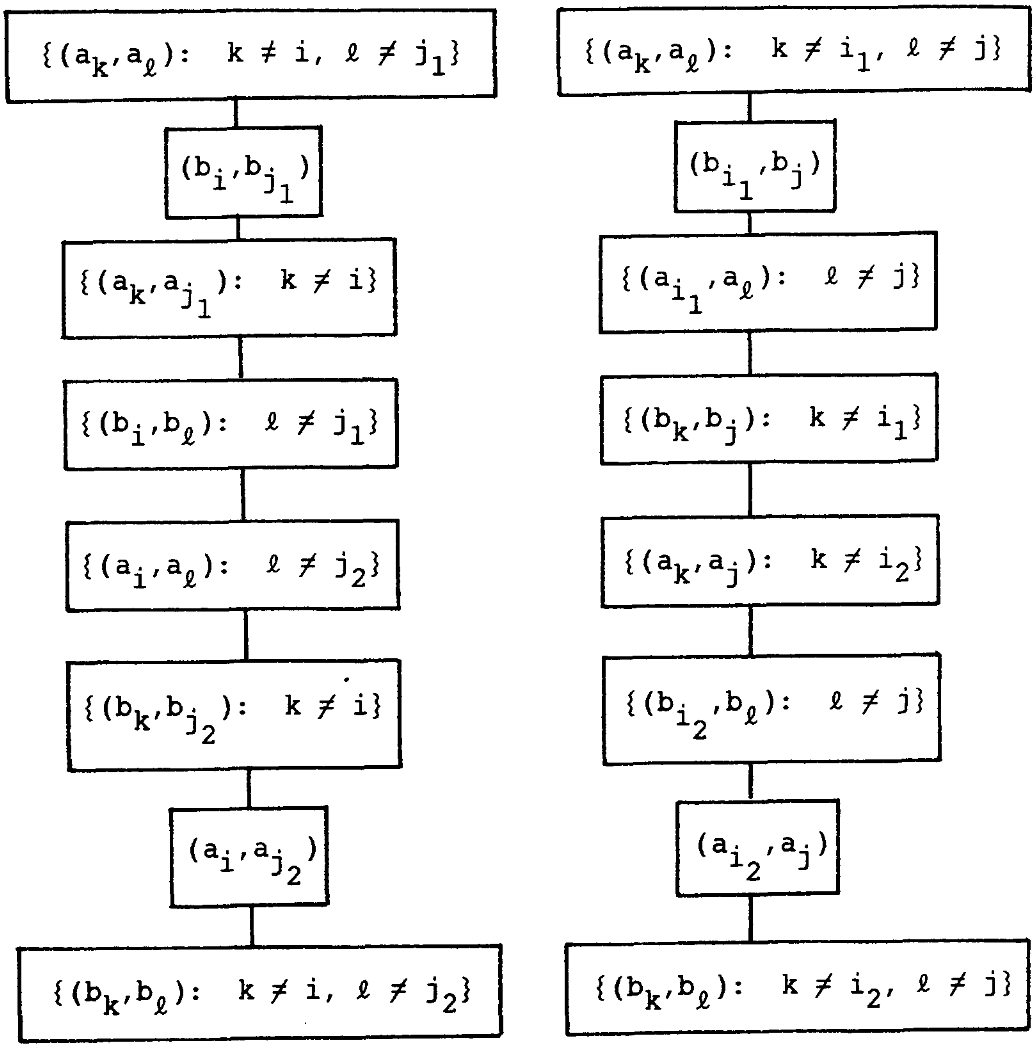


Fig. 1.

We observe that each minimal element is incomparable with $2n-1$ maximal elements, so $|N_P| = n^2(2n-1) = 2n^3 - n^2$. For each $i = 1, 2, \ldots, n$, the elements $\{((b_i, b_j), (a_i, a_l)): 1 \leq j,\ l \leq n\}$ are called the ith *row face* of N_P. Similarly, for each $j = 1, 2, \ldots, n$, the elements $\{((b_i, b_j), (a_k, a_j)): 1 \leq i,\ k \leq n\}$ are called the jth *column face* of N_P. It is easy to see that for each $i = 1, 2, \ldots, n$, there exists a linear extension of P reversing all the pairs in any row or column face. Since any nonforced pair belongs to some row or column face, this is just another way of expressing the face that $\dim P \leq 2n$. Hereafter, we will use the term *face* to mean either a row face or column face. We will say that a linear extension L *reverses a face* when all the nonforced pairs in the face are reversed in L.

Let us say that a linear extension L of P is *saturated* if there is no linear extension L' of P for which the nonforced pairs reversed by L are a proper subset



$L_R(i; j_1, j_2)$ $\qquad$ $L_C(j; i_1, i_2)$

Fig. 2.

of the nonforced pairs reversed by L'. We now proceed to show that every saturated linear extension belongs to one of two easily recognizable types. The Type 1 extensions reverse a (unique) face of N_P. For each $i = 1, 2, \ldots, n$ and each pair (j_1, j_2) with $1 \leq j_1, j_2 \leq n$, we will define a Type 1 extension $L_R(i; j_1, j_2)$ reversing the ith row face of N_p. Similarly, for each $j = 1, 2, \ldots, n$ and each pair (i_1, i_2) with $1 \leq i_1, i_2 \leq n$, we will define a Type 1 extension $L_C(j; i_1, i_2)$ reversing the jth column face. We find it convenient to specify $L_R(i; j_1, j_2)$ and $L_C(j; i_1, i_2)$ in block form. See Fig. 2.

In the definitions of $L_R(i, j_1, j_2)$ and $L_C(j; i_1, i_2)$, the ordering of the elements in the blocks is arbitrary, since with any assigned orderings, exactly the same set of nonforced pairs is reversed. Furthermore, it is obvious that each of these extensions is saturated. The following result summarizes the obvious properties of the Type 1 extensions. For convenience, we state the result in terms of rows although in subsequent arguments, we will also require the obvious dual results for columns.

Lemma 1. *Let $L_R(i; j_1, j_2)$ be a Type 1 extension. Then the following statements hold:*

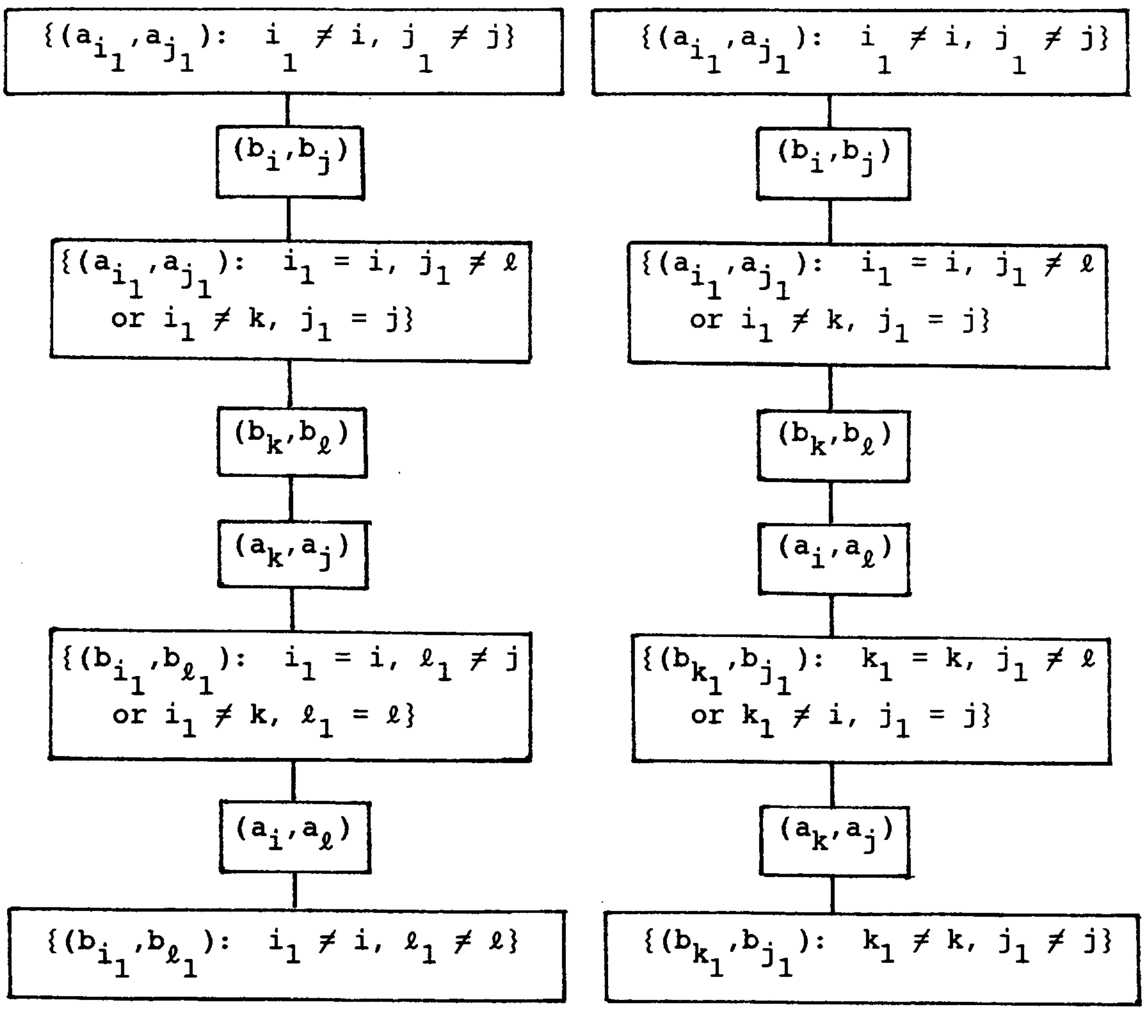
(1) *$L_R(i; j_1, j_2)$ reverses n^2+2n-2 nonforced pairs.*

(2) *$L_R(i; j_1, j_2)$ reverses no pairs in any row face except the ith row face.*

(3) *If $j_1 \neq j_2$, $L_R(i; j_1, j_2)$ reverses n nonforced pairs in each of the j_1st and j_2nd column faces and reverses one nonforced pair in each of the other $n-2$ column faces.*

(4) *If $j_1 = j_2$, $L_R(i; j_1, j_2)$ reverses $2n-1$ nonforced pairs in the j_1st column face and reverses one nonforced pair in each of the other $n-1$ column faces.*

Now, we define a second type of saturated extension. For convenience, we also describe these Type 2 extensions in block form. For each (i, j, k, l) with $i \neq k$ and $j \neq l$, we define $M_R(i, j, k, l)$ and $M_C(i, j, k, l)$ as shown in Fig. 3.



$M_R(i,j,k,\ell)$ $\qquad$ $M_C(i,j,k,\ell)$

Fig. 3.

The following result follows immediately from the definition. As before, we state the result only for rows.

Lemma 2. *Let $M_R(i, j, k, l)$ be a Type 2 saturated linear extension of P. Then the following statements hold:*

(1) *$M_R(i, j, k, l)$ reverses $4n-2$ nonforced pairs.*

(2) *$M_R(i, j, k, l)$ reverses $2n-1$ nonforced pairs in the ith row face, 1 nonforced pair in the kth row face, and no nonforced pairs in any other row face.*

(3) *$M_R(i, j, k, l)$ reverses n nonforced pairs in the jth column face, n nonforced pairs in the kth column face, and no nonforced pairs in any other column face.*

We are now ready to show that every saturated linear extension of P belongs to one of these two types.

Lemma 3. *Let L be a saturated linear extension of P. Then either L is a Type 1 or Type 2 extension.*

Proof. A saturated linear extension which reverses a face will be shown to be a Type 1 extension, and one which does not reverse a face will be shown to be a Type 2 extension. Let L be an arbitrary saturated linear extension of P. Then consider the two largest minimal elements in P. There are two cases. Either they share a common coordinate or they do not. Suppose first that the largest minimal element in L is (b_i, b_j) and the second largest minimal element is (b_k, b_j) where $i \neq k$. We then show that L is a Type 1 extension $L_C(j; i, i_1)$ for some i_1. To see this, we observe that the maximal elements $\{(a_i, a_l): 1 \leq l \leq n,\ l \neq j\}$ are under (b_i, b_j) but over (b_k, b_j). Furthermore, the maximal elements $\{(a_{i'}, a_j): 1 \leq i' \leq n\}$ are under (b_k, b_j). If we let (a_{i_1}, a_j) be the lowest maximal element in L, then the minimal elements $\{(b_{i_1}, b_l): 1 \leq l \leq n,\ l \neq j\}$ are over (a_{i_1}, a_j) but under all other maximal elements. Since the minimal elements $\{(b_k, b_j): 1 \leq k \leq n\}$ are also over (a_{i_1}, a_j), it follows easily that $L = L_C(j; i, i_1\}$. Dually, if the highest two minimal elements in L are (b_i, b_j) and (b_i, b_l) with $j \neq l$, then there is some j_1 for which $L = L_R(i; j, j_1)$.

For the second case, suppose the highest minimal elements in L is (b_i, b_j) and the second highest is (b_k, b_l) where $i \neq k$ and $j \neq l$. Then there are exactly two maximal elements under (b_i, b_j) and (b_k, b_l), namely (a_i, a_l) and (a_k, a_j). If $(a_i, a_l) <$ (a_k, a_j) in L, then it follows easily that $L = M_R(i, j, k, l)$. Similarly, if $(a_k, a_j) <$ (a_i, a_l) in L, then $L = M_C(i, j, k, l)$. $\square$

We are now ready to prove the principal result of this paper.

Theorem 2. *For each $n \geq 3$, $\dim(S_n^0 \times S_n^0) = \dim(P) = 2n-2$.*

Proof. To show that $\dim(P) \leq 2n-2$, consider the following set of $2n-2$ Type 1 extensions:

$$\mathcal{L}=\{L_R(i;n-1,n):1\leq i\leq n\}\cup\{L_C(j;1,2):1\leq j\leq n-2\}.$$

We claim that every nonforced pair is reversed in at least one extension in $\mathcal{L}$. To see that this is true we observe that every nonforced pair in any row face is obviously reversed. Similarly, we are certain that every nonforced pair in any one of the first $n-2$ column faces is reversed. This leaves only the nonforced pairs in the $(n-1)$st and nth column faces. But for each $i=1,2,\ldots,n$, the nonforced pairs in $\{((b_i,b_{n-1}),(a_k,a_{n-1})):1\leq k\leq n\}$ are reversed in $L_R(i;n-1,n)$. Similarly, for each $i=1,2,\ldots,n$, the nonforced pairs in $\{((b_k,b_n),(a_i,a_n)):1\leq k\leq n\}$ are reversed in $L_R(i;n-1,n)$. Thus all nonforced pairs are reversed, and $\dim(S_n^0\times S_n^0)\leq 2n-2$. (The reader should note that in the definition of $\mathcal{L}$, the choice of $(i_1,i_2)=(1,2)$ in the extensions $L_C(j;i_1,i_2)$ was arbitrary. Any extension reversing the jth column face would suffice.)

To show that $\dim(S_n^0\times S_n^0)\geq 2n-2$, we assume to the contrary that $\dim(S_n^0\times S_n^0)=t\leq 2n-3$ and choose a collection $\mathsf{L}=\{L_1,L_2,\ldots,L_t\}$ of linear extensions of P reversing all nonforced pairs. Clearly, we may assume that each extension in L is saturated. We adopt the terminology of saying that the ith row face is *used* in L if there is some $L_\alpha\in\mathsf{L}$ with either $L_\alpha=L_R(i;j_1,j_2)$ or $L_\alpha=M_R(i,j,k,l)$. Otherwise, we say that the ith row face is *unused.* Dual statements apply to column faces.

Since no extension in L reverses more than n^2+2n-2 nonforced pairs and there are $n^2(2n-1)$ pairs to be reversed, we know that

$$t\geq\left\lceil\frac{2n^3-n^2}{n^2+2n-2}\right\rceil\geq 2n-5.$$

Now suppose that there are four (or more) unused faces. Choose four of them. Then there are $4n^2$ nonforced pairs to be reversed in these faces if all four are row faces or if all four are column faces. If three are row faces and one is a column, or three columns and one row, then there are $4n^2-3$ pairs to be reversed. Finally, if two are rows and two are columns, then there are $4n^2-4$ pairs to be reversed. However, we observe from Lemma 1, that a Type 1 extension reverses a total of at most $2n+2$ nonforced pairs on faces other than the one it reverses. From Lemma 2, we observe that a Type 2 extension reverses exactly $2n+1$ pairs on faces other than the one it uses. Thus, we must have $t(2n+2)\geq 4n^2-4$, and thus

$$t\geq\frac{4n^2-4}{2n+2}=2n-2.$$

The contradiction shows that all but at most three of the $2n$ row and columns faces must be used. We conclude that $t=2n-3$ and that there are exactly three unused faces.

Suppose first that there are three column faces which are unused. Then there are $3n^2$ nonforced pairs to be reversed in these three faces. However, any Type 1 extension in L reversing a column face contributes nothing to this effort while a Type 2 extension using a column face reverses at most 1 nonforced pair on the unused column faces. On the other hand, the n extensions using row faces each reverse at most $2n+1$ pairs among the three unused column faces. This requires

$$(n-3)1+n(2n+1)\geqslant 3n^2,$$

which is a contradiction.

Next, suppose that there are two unused row faces and one unused column face. Then there are $2n^2$ nonforced pairs on the unused row faces. The Type 1 extensions reversing row faces contribute nothing towards this effort, and the Type 2 extensions using row faces reverse at most 1 nonforced pair on the unused row faces. On the other hand, the extensions using column faces each reverse at most $2n$ nonforced pairs on the unused row faces. This requires

$$(n-2)1+(n-1)2n\geqslant 2n^2,$$

which is also a contradiction. The argument when there are two unused column faces and one unused row face is dual. With this observation the proof is complete. $\square$

3. Concluding remarks

Despite the specialized nature of the result proved in the preceding section, it can be shown that the basic approach is valid under the assumption that Conjecture 1 is true. Specifically, if there exist partial orders P and Q for which $\dim(P\times Q)$ is significantly less than $\dim(P)+\dim(Q)$, then such partial orders can be found among the height one partial orders. This may be deduced from the observation made by Kimble [6] that if P is a partial order of arbitrary height and C is a sufficiently long chain, then there is a height one partial order P' contained in $P\times C$ with

$$\dim(P)\leqslant\dim(P')\leqslant 1+\dim(P).$$

The partial order P' is called the *horizontal split* of P (see [12] for details). Furthermore, it is easy to see that for height one partial orders P and Q, we can restrict our attention to reversing the nonforced pairs of $P\times Q$ which involve incomparable min-max pairs. This amounts to computing the interval dimension but for height one posets, this invariant differs from dimension by at most one [11].

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