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NOTE

A NOTE ON RANKING FUNCTIONS

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In this issue, W.J. Walker introduces the lattice $L(n, r)$ as the set of all possible results when n competitors are matched in a series of r races. A result is an r-term nondecreasing sequence of integers selected from $\{1, 2, \ldots, n\}$. The dimension of $L(n, r)$ is at most r since it is a subposet of $\boldsymbol{R}^r$. Walker conjectures that $L(n, r)$ is in fact the intersection of r consistent linear extensions and verifies this conjecture when $r \leqslant 2$ as well as for the case $(n, r) = (4, 3)$. In this note, we show that the general conjecture does not hold by proving that for every $r \geqslant 3$ and every $t \geqslant r$, there exists an integer n_0 so that if $n \geqslant n_0$, then $L(n, r)$ is not the intersection of t consistent linear extensions.

In this note, we follow the notation and terminology of the article by Walker which appears in this issue [2]. We let $L(n, r)$ denote the set of all r-term nondecreasing sequences with entries from $\{1, 2, \ldots, n\}$. These sequences are called *results* and represent the score sequences which arise when n competitors are matched in a series of r races. There is a natural partial order on $L(n, r)$ defined by $(x_1, x_2, \ldots, x_r) \geqslant (y_1, y_2, \ldots, y_r)$ if and only if $x_i \leqslant y_i$ in $\boldsymbol{R}$ for all $i = 1, 2, \ldots, r$. With this ordering, the largest element in $L(n, r)$ is the r-term vector $(1, 1, \ldots, 1)$ which corresponds to a first place finish in every race.

Recall that the *dimension* of a finite poset P is the least s so that it is possible to assign to each $x \in P$ a vector $(x_1, x_2, \ldots, x_s)$ of real numbers so that $x \leqslant y$ in P if and only if $x_i \leqslant y_i$ in $\boldsymbol{R}$ for $i = 1, 2, \ldots, s$. Equivalently, the dimension of P is the least s for which P is the intersection of s linear extensions. It is obvious that the dimension of a poset and its dual are the same, so the dimension of $L(n, r)$ is at most r. We refer the reader to the survey article [1] for extensive background material on the dimension of posets.

For a result $R \in L(n, r)$, let $S(R)$ be the multiset consisting of the entries of the vector R. In what follows, when we take the union of multisets, we mean that

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repetitions are to be counted. For example, if $R=(1,1,2,3,3,3)$, then $S(R)=\{1,1,2,3,3,3\}$, and $\{1,1,2,3\}\cup\{1,3,3,3\}=\{1,1,1,2,3,3,3,3\}$.

Following Walker, we define a linear extension M of $L(n, r)$ to be *consistent* when there does not exist an integer $p \geq 2$, a result $R \in L(n, pr)$ and two families $\{A_i: 1 \leq i \leq p\}$ and $\{B_i: 1 \leq i \leq p\}$ of results from $L(n, r)$ so that:

(1) $A_i > B_i$ in M for $i = 1, 2, \ldots, p$, and

(2) $S(R) = \bigcup \{S(A_i): 1 \leq i \leq p\} = \bigcup \{S(B_i): 1 \leq i \leq p\}$.

Roughly speaking, a consistent linear extension is an ordering of $L(n, r)$ which can arise if prize money is paid on the basis of the competitors' finishing positions in the series of races.

Walker shows that $L(n, r)$ is the intersection of the set of all consistent linear extensions, so we may define the *consistent dimension* of $L(n, r)$ as the least t for which $L(n, r)$ is the intersection of t consistent linear extensions. Walker shows that the consistent dimension of $L(n, r)$ equals the dimension of $L(n, r)$ when $r \leq 2$ and when $(n, r) = (4, 3)$. The principal result here will be to show that in general, the consistent dimension of $L(n, r)$ is much larger than its dimension.

Theorem. *For every $r \geq 3$ and every $t \geq r$, there exists an integer n_0 (depending on r and t) so that if $n \geq n_0$, then the consistent dimension of $L(n, r)$ is larger than t.*

Proof. We present the argument when $t = 3$. The extension to larger values of t is immediate. Suppose that $L(n, 3)$ is the intersection of t linear extensions $M_1, M_2, \ldots, M_t$. We show that if n is sufficiently large in comparison to t, then there is at least one $\alpha \in \{1, 2, \ldots, t\}$ for which M_α is not consistent. The argument uses Ramsey's theorem.

Consider a 6-element subset $\{i_1 < i_2 < i_3 < i_4 < i_5 < i_6\}$ of $\{1, 2, \ldots, n\}$. This subset determines a special pair of incomparable elements of $L(n, 3)$, namely (i_2, i_3, i_6) and (i_1, i_4, i_5). It follows that we may choose some $\alpha \in \{1, 2, \ldots, t\}$ so that $(i_2, i_3, i_6) > (i_1, i_4, i_5)$ in M_α. This choice function defines a mapping of the 6-element subsets of $\{1, 2, \ldots, n\}$ to be the t-element set $\{1, 2, \ldots, t\}$.

It follows from Ramsey's theorem that if n is sufficiently large in comparison to t, then there is an element $\alpha \in \{1, 2, \ldots, t\}$ and a 10-element subset $H = \{i_1 < i_2 < \cdots < i_{10}\}$ of $\{1, 2, \ldots, n\}$ so that all 6-element subsets of H are mapped to α. In particular, this means that there is some M_α in which the special pairs of incomparable elements determined by the subsets

$$S_1 = \{i_1 < i_2 < i_3 < i_4 < i_5 < i_6\},$$

$$S_2 = \{i_3 < i_4 < i_5 < i_6 < i_7 < i_8\},$$

and

$$S_3 = \{i_3 < i_4 < i_7 < i_8 < i_9 < i_{10}\}$$

are in the following order in the linear extension M_α:

$$(i_2, i_3, i_6) > (i_1, i_4, i_5),$$

$$(i_4, i_5, i_8) > (i_3, i_6, i_7),$$

and

$$(i_4, i_7, i_{10}) > (i_3, i_8, i_9).$$

Now we use the fact that M_α is a linear extension of $L(n, 3)$ to conclude that the following statements hold for M_α:

$$A_1 = (i_1, i_3, i_6) > (i_2, i_3, i_6) > (i_1, i_4, i_5) = B_1,$$

$$A_2 = (i_3, i_5, i_8) > (i_4, i_5, i_8) > (i_3, i_6, i_7) = B_2,$$

and

$$A_3 = (i_4, i_7, i_9) > (i_4, i_7, i_{10}) > (i_3, i_8, i_9) = B_3.$$

Taking multiset unions, we observe that:

$$\{i_1, i_3, i_3, i_4, i_5, i_6, i_7, i_8, i_9\} = A_1 \cup A_2 \cup A_3 = B_1 \cup B_2 \cup B_3.$$

This shows that M_α is not consistent. With this observation, the proof is complete. □

References

[1] D. Kelly and W.T. Trotter, Dimension theory for ordered sets, in: I. Rival, ed., Ordered Sets, (Reidel, Dordrecht, 1982) 171–211.

[2] W.J. Walker, Ranking functions and axioms for linear orders, Discrete Math. 67 (1987) 299–306.