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Problems and Conjectures in the Combinatorial Theory of Ordered Sets

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Dedicated to the memory of G. A. Dirac

49 questions from the theory of ordered sets are stated, and the present knowledge about each is surveyed.

1 Introduction

The purpose of this article is to survey a number of combinatorial problems and conjectures for ordered sets. This area of combinatorial mathematics is relatively new and is experiencing rapid growth. Accordingly, it is impossible to claim that the article is exhaustive in scope. Rather, we concentrate on problems which fall under the heading of "extremal problems". Furthermore, we have chosen problems which exhibit the single most attractive feature of combinatorial mathematics in that the problems can be understood by nonspecialists. Throughout the paper, we consider an *ordered set* as a pair (X, P) where X is a set and the *partial* order P is a reflexive, antisymmetric and transitive relation on X.

2 Dimension problems

The *dimension* of an ordered set (X, P), denoted $\dim(X, P)$, is the least t for which P is the intersection of t linear orders on X. Our first conjecture is one of the best known problems in dimension theory.

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Conjecture 1. *If (X,P) is any ordered set with $|X| \geq 3$, then there exist distinct points $x,y \in X$ so that*

$$\dim(X,P) \leq 1 + \dim(X - \{x,y\},\ P(X - \{x,y\})).$$

This conjecture arises from attempts to provide a simple inductive proof of the following basic inequality due to T. Hiraguchi [13]: $\dim(X,P) \leq |X|/2$ when $|X| \geq 4$. We believe the first published reference to the conjecture is [3], but the problem is so natural that one cannot be certain. See Kelly's note [16] for additional background information on this problem.

Conjecture 2. *For every pair m,n of positive integers, there exist ordered sets (X,P) and (Y,Q) so that*

$$\dim((X,P) \times (Y,Q)) = \max\{\dim(X,P), \dim(Y,Q)\}.$$

This problem comes from the following elementary inequalities governing cartesian products of ordered sets:

$$\max\{\dim(X,P), \dim(Y,Q)\} \leq \dim((X,P) \times (Y,Q)) \quad (1)$$

$$\dim((X,P) \times (Y,Q)) \leq \dim(X,P) + \dim(Y,Q) \quad (2)$$

In many cases, the second inequality is tight. For example, K. Baker [2] showed that this occurs whenever the two ordered sets are nontrivial and have greatest and least elements. However, very little is known about the accuracy of the first inequality. In [34], Trotter showed that for each $n \geq 3$, there exists an n-dimensional, ordered set (X,P) with the property that $\dim((X,P) \times (X,P)) = 2n - 2$. This ordered set is the *standard* example, i.e., the family of all 1-element and $(n-1)$-element subsets of an n-element set ordered by inclusion. Perhaps it is true that for each $n \geq 1$, there exists an ordered set (X,P) so that $\dim(X,P) = n$ and $\dim((X,P) \times (X,P)) = n$.

The standard example figures in another intriguing dimension theory problem.

Problem 3. *For each $n \geq 1$, determine the least integer $f(n)$ so that if (X,P) is any ordered set in which each point is comparable with at most n other points, then $\dim(X,P) \leq f(n)$.*

Z. Füredi and J. Kahn [10] have supplied a clever argument to show that there exists an absolute constant c so that $f(n) \leq c\,n\,(\log n)^2$. On the other hand, the standard example provides the trivial lower bound $f(n) \geq n+1$. Although no better bound is known, it appears likely that $f(n)/n$ tends to infinity.

An ordered set (Y, Q) is called a *homeomorph* of (X, P) if the diagram for (Y, Q) is obtained from the diagram of (X, P) by inserting points on edges.

Problem 4. *Is it true that if* (Y, Q) *is a homeomorph of* (X, P)*, then* $\dim(X, P) \leq \dim(Y, Q) \leq 2\dim(X, P)$*?*

There are many analogies between dimension for ordered sets and chromatic number of graphs. Here is a problem which comes from attempts to see how far this analogy extends. In order for a graph to have large chromatic number, it must contain a relatively small subgraph which has relatively large chromatic number. To be somewhat more precise, Kierstead, Szemerédi and Trotter [21] proved that if G is a graph on n vertices, and G has no subgraph H whose chromatic number exceeds c and whose radius in G is at most $2kn^{1/k}$, then the chromatic number of G is at most $k(c-1)+1$. The special case of this theorem when $c = 2$ was conjectured by P. Erdös.

Problem 5. *What is the maximum value* $f(n, m, k)$ *of the dimension of an ordered set on* n *points in which no subordered set on* m *points has dimension exceeding* k*?*

A special case of this problem may be of particular interest.

Problem 6. *What is the maximum value* $f(n, k)$ *of the dimension of an ordered set on* n *points which does not contain a standard example of dimension* k*?*

A solution to the preceding problem may shed some light on the following problem which is due to B. Sands.

Problem 7. *What is the least integer* $f(n)$ *for which there exists a lattice of dimension* n *having* $f(n)$ *points?*

Of course, the distributive lattice 2^n consisting of all subsets of an n-element set ordered by inclusion has dimension n so $f(n) \leq 2^n$. But in [12], B. Ganter, P. Nevermann, K. Reuter and J. Stahl observed that the lattice of subspaces of the projective plane of order 3 has 28 points and dimension at least 5. More generally, they showed that $f(n) < c^n$ for every $c > 1$. This result follows from their observation that if Π_n denotes the lattice of all partitions of an n-element set, then there exist absolute constants c_1, c_2 so that $c_1 n^2 < \dim(\Pi_n) < c_2 n^2$. Perhaps there is an exact answer for the dimension of the lattice of partitions of a set. At least it should be possible to determine the limit of $\dim(\Pi_n)/n^2$. As far as I know, the dimension of the lattice of partitions of an integer has not been investigated.

These same authors also show that the lattice of subspaces of an n-dimensional linear vector space over $GF(2)$ also provides a subexponential bound on $f(n)$. For this lattice, they show that the dimension is at least $2^n/n$ and at most 2^n. It should be possible to obtain a good estimate for the dimension of this lattice.

Recently, Z. Füredi and J. Kahn [11] observed that a much better bound on the minimum size of an n-dimensional lattice can be derived from the lattice obtained by adding a zero and a one to the ordered set formed by the points and the lines of a finite projective plane. They observed that if the order of the projective plane is n, then the dimension of the lattice is at least εn where ε is an absolute constant. This shows that $f(n) < cn^2$. Probably the exponent is right in this estimate.

The following problem is due to J. Ginsburg. Define the *cutset number* of (X, P) as the least nonnegative integer n so that for each $x \in X$, there exists a subset S of at most n points with each point of S incomparable to x so that every maximal chain contains a point from $S \cup \{x\}$.

Problem 8. *Does there exist a function* $f(n)$ *so that* $\dim(X, P) \leq f(n)$ *whenever the cutset number of* (X, P) *is* n*?*

3 Problems on linear extensions

Let $E(P)$ denote the set of all linear extensions of a partial order P on a finite set X. For a distinct pair $x, y \in X$, define $\mathrm{PROB}(x < y)$ as the ratio of the number of extensions in $E(P)$ which place x before y divided by the total number of extensions in $E(P)$. The following beautiful conjecture was made by M. Fredman [9]:

Conjecture 9. *If* (X, P) *is not a chain, there exist distinct points* $x, y \in X$ *so that* $1/3 \leq \mathrm{PROB}(x < y) \leq 2/3$.

Linial [25] proved that the conjecture holds if the width of (X, P) is at most two. Aigner [1] classified the width two ordered sets for which the inequality is tight. However, the best result to date for arbitrary width is due to Kahn and Saks [15]. They have shown that there exists x, y so that $3/11 \leq \mathrm{PROB}(x < y) \leq 8/11$. They also make the following conjecture.

Conjecture 10. *For every $\varepsilon > 0$, there exists an integer n so that if (X, P) is any ordered set with $width(X, P) \geq n$, then there exist distinct points $x, y \in X$ so that $1/2 - \varepsilon \leq \text{PROB}(x < y) \leq 1/2 + \varepsilon$.*

Additional conjectures of this general flavor are given in Saks' note [31].

There are a number of interesting problems concerning the cardinality of $E(P)$. The following problem is relayed by J. Kahn.

Problem 11. *Does there exist an efficient algorithm which accepts an ordered set (X, P) as input and which outputs a number m satisfying that $m \leq |E(P)| \leq m\,2^n$, where $n = |X|$?*

Problem 12. *For integers n, k with $0 \leq k \leq \binom{n}{2}$, find the greatest integer $f(n, k)$ for which there exists an ordered set (X, P) with $n = |X|$, $|E(P)| = f(n, k)$, and with exactly k of the $\binom{n}{2}$ pairs of X comparable in P.*

Some information is known about the extremal ordered sets. It is relatively easy to establish that if (X, P) is an ordered set with n points and k comparable pairs and $|E(P)| = f(n, k)$, then there exists a function g from X to the set R of real numbers so that $x < y$ in P if and only if $g(x) + 1 < g(y)$. Such ordered sets are called *semiorders* (also *unit interval orders*). Here is a variant of the preceding problem posed by I. Rival.

Problem 13. *For integers n, k with $0 \leq k \leq n^2/4$, find the greatest integer $g(n, k)$ for which there exists an ordered set (X, P) with $|X| = n$, $|E(P)| = g(n, k)$, and with k edges in the diagram of (X, P).*

It may also be of interest to find those ordered sets with a specified number of comparable pairs or a specified number of edges in the diagram for which the number of linear extensions is minimum.

Let (X, P) be an ordered set. For each $L \in E(P)$, let $\text{jump}(L; P)$ count the number of pairs of elements which are consecutive in L and incomparable in P. Then define the *jump number* of (X, P), denoted $\text{jump}(X, P)$, by $\text{jump}(X, P) = \min\{\text{jump}(L; P) : L \in E(P)\}$. Call (X, P) *jump critical* if $\text{jump}(X - \{x\},\ P(X - \{x\})) < \text{jump}(X, P)$ for every $x \in X$.

Problem 14. *For each integer $k \geq 1$, find the least integer $f(k)$ so that if $\text{jump}(X, P) = k$ and (X, P) is jump critical, then $|X| \leq f(k)$.*

El–Zahar and Schmerl [6] show that $f(k) \leq (k + 1)!$.

Perhaps this inequality is far from best possible. The best known lower bound comes from examples constructed by El–Zahar and Rival [5] which yield $f(k) \geq ck$ where $3 < c < 4$.

Let J_n denote the ordered set consisting of the linear sum of n two element antichains, i.e., points from distinct antichains are comparable. Then let A_n denote an n-element antichain. J. Schmerl [32] has proved that an ordered set with large jump number either contains a large J_n or a large A_n.

Problem 15. *For positive integers m and n, find the least integer $f(m,n)$ so that if (X,P) is any ordered set with* $\mathrm{jump}(X,P) \geq f(m,n)$, *then either (X,P) contains J_m or A_n.*

4 Interval orders and generalizations

Let C be a collection of distinct closed intervals of the real line. Define a partial order P on C by $A < B$ in P if and only if $a < b$ on the real line for every $a \in A$ and every $b \in B$. An ordered set isomorphic to one obtained by this definition is called an *interval order.*

Let $f(n)$ be the least integer so that if (X,P) is an interval order and $\mathrm{length}(X,P) = n$, then $\dim(X,P) \leq f(n)$. Rabinovitch [29] proved the existence of $f(n)$ and Bogart, Rabinovitch and Trotter [4] showed that there exists a constant $c > 2$ so that $\log\log n \leq f(n) < \log_c n$. The upper bound has been improved by Rödl and Trotter [30] to $f(n) < (\log\log n)^c$ for some c. Probably $f(n) < (\log\log n)^{1+\epsilon}$ and perhaps $f(n) < c\log\log n$.

However, there has been little work done on the dual problem.

Problem 16. *Find the least $f(n)$ so that if (X,P) is an interval order of width n, then* $\dim(X,P) \leq f(n)$.

Call an ordered set (X,P) *dimension critical* if, for every $x \in X$, $\dim(X - \{x\}, P(X - \{x\})) < \dim(X,P)$.

Problem 17. *How many dimension critical interval orders are there on n points?*

Problem 18. *How many dimension critical interval orders of dimension n are there?*

Let $I(n)$ denote the interval order determined by the set of nondegenerate closed intervals with integer endpoints from $\{1, 2, \cdots, n\}$.

Problem 19. *Does there exist a function* $f(n)$ *so that if* (X,P) *is an interval order and* $\dim(X,P) > f(n)$, *then* (X,P) *contains* $I(n)$*?*

The *interval count* of an interval order is the minimum number of different length intervals required in a representation. P. Fishburn [7] discusses many intriguing problems for interval count and we mention here two of them.

Conjecture 20. *If* (X,P) *is an interval order and* X *contains* $3n$ *points, then the interval count of* (X,P) *does not exceed* n.

The interval order which is defined by $\{[-2i-1, 2i+1] : 0 \le i \le n\} \cup \{[-2i-1, -2i] : 1 \le i \le n\} \cup \{[2i, 2i+1]; 1 \le i \le n\}$ shows that this conjecture if true is best possible.

Problem 21. *Does there exist a constant c so that the removal of a point from an interval order decreases the interval count by at most c?*

Consider an angle in the Euclidean plane as determining an infinite region bounded by two incident rays. An ordered set isomorphic to a family of regions formed by angles in the plane and ordered by set inclusion is called an *angle order.* Fishburn and Trotter [8] proved that every interval order is an angle order and that every ordered set with dimension at most four is an angle order. They also proved that there exists an ordered set of dimension 7 which is not an angle order.

Problem 22. *Find the least integer t for which there exists an ordered set of dimension t which is not an angle order.*

Problem 23. *If* (X,P) *is an angle order, is the ordered set formed by adding a zero to* (X,P) *also an angle order?*

Call an ordered set a *circle order* if it is isomorphic to a family of circular regions in the plane ordered by inclusion.

Problem 24. *Is every three dimensional ordered set a circle order?*

Problem 25. *Find the least integer t for which there exists an ordered set of dimension t which is not a circle order.*

5 Ramsey theoretic problems

Let (X,P) and (Y,Q) be ordered sets. We write $(Y,Q) \to (X,P)$ when every partitioning of the points of Y into 2 classes yield a subset $X' \subset Y$

so that $(X', Q(X'))$ is isomorphic to (X, P) and all points of X' belong to the same class.

Problem 26. *For each $n \geq 2$, find the least integer $f(n)$ so that if (X, P) is an ordered set of width at most n, then there exists an ordered set (Y, Q) of width at most $f(n)$ so that $(Y, Q) \to (X, P)$.*

It is trivial to see that $2n - 1 \leq f(n) \leq n^2$. In [23], H. Kierstead and W. Trotter show that $f(n) \geq 2n$. The argument for this bound seems to leave room for improvement, and I suspect that $f(n) > \varepsilon\, n^2$ for some constant ε.

Problem 27. *For each $n \geq 1$, let $g(n)$ be the least positive integer so that if (X, P) is any ordered set with $|X| = n$, then there exists an ordered set (Y, Q) with $|Y| = g(n)$ so that $(Y, Q) \to (X, P)$.*

Again, we start with the trivial bounds $2n - 1 \leq g(n) \leq n^2$. But in this case, we can at least improve [23] to $n^2/4 \leq g(n) \leq n^2 - n + 1$. The reader may naturally ask why we do not pose the analogous problems for dimension and length. The answer is that these problems have been completely solved by Nešetřil and Rödl [26]. They proved that if $\dim(X, P) = n$, then there exists (Y, Q) with $\dim(Y, Q) = n$ so that $(Y, Q) \to (X, P)$. Furthermore, if $\text{length}(X, P) = n$, there exists (Y, Q) with $\text{length}(Y, Q) = 2n - 1$ so that $(Y, Q) \to (X, P)$. Clearly, these results are best possible.

It follows immediately from Ramsey's theorem that if (X, P) is an interval order, then there exists an interval order (Y, Q) so that $(Y, Q) \to (X, P)$.

Problem 28. *If (X, P) is an interval order on n points, what is the least $f(n)$ so that there exists an interval order (Y, Q) so that $(Y, Q) \to (X, P)$ and the cardinality of Y is at most $f(n)$?*

Problem 29. *If (X, P) is an interval order on n points and has length (respectively width, dimension) k, what is the least $f(n, k)$ so that there exists an interval order (Y, Q) so that $(Y, Q) \to (X, P)$ and the length (respectively width, dimension) of (Y, Q) is at most $f(n, k)$? In particular, does $f(n, k)$ depend on n?*

On the other hand, it is relatively easy to see that there exists an angle order (X, P) for which there is no angle order (Y, Q) so that $(Y, Q) \to (X, P)$.

Problem 30. *If (X, P) is a circle order, does there exist a circle order (Y, Q) so that $(Y, Q) \to (X, P)$?*

Define the *angle dimension* of (X, P), denoted $\text{Adim}(X, P)$, as the least t for which there exists t binary relations $A_1, A_2, \cdots, A_t$ on X so that $P = A_1 \cap A_2 \cap \cdots \cap A_t$ and (X, A_i) is an angle order for $i = 1, 2, \cdots, t$.

Problem 31. *Find the least* $f(n,t)$ *so that if* $|X| = n$, $\text{Adim}(X, P) = t$, *and* $x \in X$, *then* $\text{Adim}(X - \{x\},\ P(X - \{x\})) \geq t - f(n,t)$. *In particular, does there exist a constant c so that* $f(n,t) \leq c$ *for all* n, t*?*

Problem 32. *If* $|X| = n$, *what is the maximum value of* $\text{Adim}(X, P)$*?*

Problem 33. *Does there exist a function* $f(n)$ *so that if* $\text{Adim}(X, P) = n$ *then there exists* (Y, Q) *with* $\text{Adim}(Y, Q) \leq f(n)$ *so that* $(Y, Q) \to (X, P)$*?*

Of course, we can also define the *circle dimension* of an ordered set in the obvious fashion and rephrase the last three problems in terms of this parameter. Needless to say, the concept of dimension is one that admits an endless number of variations. Some of these have been explored, and perhaps more work on this theme is worthwhile.

6 Problems from recursive combinatorics

Consider the following game theoretic setting involving two players A and B, a class C of ordered sets and an integer t. It is further assumed that if $(X, P) \in C$, then $X = \{1, 2, \cdots, m\}$ where m is some positive integer. The players alternate moves with Player A constructing an ordered set from C and Player B partitioning this ordered set into chains. At Round i, where $i \leq m$, Player A provides the binary relation determined by the restriction of P to $\{1, 2, \cdots, i\}$. After receiving this information, Player B makes an irrevocable assignment of i to one of t sets $C_1, C_2, \cdots, C_t$ each of which is a chain in (X, P).

The game terminates and A is the winner if at some step, say Round i with $i \leq m$, Player B has no admissible move, i.e., for each $j = 1, 2, \cdots, t$, the assignment of i to C_j produces a set which is no longer a chain. If Player B is able to make admissible assignments to chains at each Round i for $i = 1, 2, . \cdots, m$, then Player B is the winner.

We say that the class C can be recursively partitioned into t chains if there exists a strategy for Player B which will enable him to defeat Player A regardless of the strategy followed by Player A.

Problem 34. *What is the least* $f(n)$ *for which the class of finite ordered sets of width n can be recursively partitioned into* $f(n)$ *chains?*

It is not at all clear that $f(n)$ exists. In a beautiful paper [19] H. Kierstead showed that $f(n) \leq (5^n - 1)/4$. The best lower bound is due to Szemerédi (see [17]), $f(n) \geq \binom{n+1}{2}$. It is of particular interest to decide whether $f(n)$ is polynomial or exponential. This problem is likely to be difficult since even the precise value of $f(2)$ is not known. It is either 5 or 6.

Kierstead and Trotter [22] have shown that the class $I(n)$ of interval orders of width n can be recursively partitioned into $3n-2$ chains. This result is best possible.

In an obvious manner, we can speak of recursively partitioning a class of ordered sets into antichains, recursively coloring a class of graphs, etc. The antichain partitioning problem has been completely solved. J. Schmerl (see [18]) proved that the class $L(n)$ of ordered sets of length n can be recursively partitioned into $\binom{n+1}{2}$ antichains and Szemerédi [18] has shown that the result is best possible.

Problem 35. *Does there exist a function $f(n)$ so that the collection of all comparability graphs in which the maximum size of an independent set is n can be partitioned into $f(n)$ complete subgraphs?*

Some nice results have been obtained for recursive dimension. H. Kierstead, G. McNulty and W. Trotter [20] showed that the family of width three ordered sets does not have finite recursive dimension. The difficulty comes from the class of height one ordered sets which are called *crowns.* These ordered sets contain k maximal elements $a_1, a_2, \cdots, a_k$ and k minimal elements $b_1, b_2, \cdots, b_k$ with $a_i > b_i$ and $a_i > b_{i+1}$ (cyclically) with $k \geq 3$. Crowns are three dimensional ordered sets and they are dimension critical. They play a key role in a number of combinatorial problems for ordered sets.

Problem 36. *What is the least $f(n)$ so that the class of width n crown-free ordered sets has recursive dimension at most $f(n)$?*

A double exponential upper bound and an exponential lower bound for $f(n)$ is established in [20]. The proof techniques strongly support the following conjecture.

Conjecture 37. *For every $k \geq 3$ and every $n \geq k$, the class of ordered sets of width n which do not contain a crown on $2m$ points for any m with $3 \leq m \leq k$ does not have finite recursive dimension.*

We refer the reader to Kierstead's survey article [18] for additional problems from recursive combinatorics.

7 Planarity

An ordered set is p
plane without edge

Problem 38. *Whi
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Problem 39. *Whi
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Some examples
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Problem 40. *Do
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planar if and only i
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Problem 41. *Doe
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Problem 42. *Doe
ordered set associa
chromatic number c*

7 Planarity

An ordered set is *planar* if it is possible to draw a diagram for it in the plane without edge crossings.

Problem 38. *Which ordered sets are planar? Develop a good algorithm for testing an ordered set for planarity.*

It makes good sense to extend this problem to the class of oriented graphs which do not contain directed cycles. Just as is done with ordered sets, we can use diagrams for acyclic oriented graphs in which the orientation of edges is indicated by the orientation in the plane. In this way, an acyclic oriented graph can be nonplanar when the underlying undirected graph is planar.

Problem 39. *Which acyclic oriented graphs are planar? Provide an efficient algorithm which tests planarity. Also characterize planarity for acyclic oriented graphs by providing a list of forbidden subgraphs.*

Some examples of forbidden subgraphs are given in [35]. One of the original motivations for studying planarity for ordered sets comes from dimension theory. A planar ordered set with greatest and least elements has dimension at most two. Trotter and Moore [36] showed that a planar ordered set with a least (or greatest) element has dimension at most three. However, D. Kelly [17] constructed planar ordered sets of arbitrary dimension.

Problem 40. *Do there exist dimension critical planar ordered sets of arbitrary dimension?*

I have just learned from W. Schnyder of the following beautiful result he has recently proved. With a graph $G = (V, E)$, we associate an ordered set (X, P) defined by $X = V \cup E$ with $x < e$ in P if and only if the vertex x is an endpoint of the edge e. Schnyder proved [33] that the graph G is planar if and only if the dimension of the associated ordered set is at most three.

Problem 41. *Does there exist a function $f(n)$ so that if G is a graph of genus n, then the dimension of the associated ordered set is at most $f(n)$?*

Problem 42. *Does there exist a function $g(n)$ so that if (X, P) is the ordered set associated with the graph G, and $\dim(X, P) = n$, then the chromatic number of G is at most $g(n)$?*

The preceding problem is related in spirit to the famous problem involving the existence of graphs with bounded maximum clique size and arbitrarily large chromatic number. The reverse problem is easier since it is easy to show that there exists an absolute constant c so that if (X, P) is the ordered set associated with graph G, and the chromatic number of G is n, then $\dim(X, P) < c \log \log n$. Apart from the value of the constant c, this result is best possible. This is shown by the complete graph on n vertices.

8 Miscellaneous problems

We have not included in this paper any Sperner theory problems only because in most cases, the formulation of these problems requires additional background material. However, we must comment that there are a large number of important problems in this area and we refer the reader to Griggs' survey article [13] for a sampling. For similar reasons, we have not discussed problems from combinatorial lattice theory.

There are a tremendous number of challenging and easily accessible problems dealing with families of subsets of a set. Here is one of my favorites. It comes from P. Frankl.

Problem 43. *Prove that there exists an absolute constant* ε *so that whenever* F *is a family of sets closed under unions, i.e.,* $A \cup B \in F$ *for every* $A, B \in F$, *then there is an element which belongs to at least* $\varepsilon|F|$ *of the sets in* F.

Actually, Frankl conjectures that the value $\varepsilon = 1/2$ works. This value is best possible as the family of all subsets of a set shows.

Here is another posed by D. Kleitman.

Problem 44. *How many colors are required to assign colors to the subsets of* $\{1, 2, \cdots, n\}$ *so that for each* $i = 1, 2, \cdots, n$, *the family* F_i *of subsets assigned color* i *is completely union free, i.e., if* $A_1, A_2, \cdots, A_k, A_{k+1} \in F_i$, *then* $A_1 \cup A_2 \cup \cdots \cup A_k \neq A_{k+1}$*?*

It is easy to see that $n + 1$ colors suffice since we can assign a set A its cardinality as a color. An easy inductive argument shows that the number of colors required is at least $n/2$.

Call a graph G a *cover graph* if there is a drawing of G in the plane which yields the diagram of an ordered set. It is obvious that a cover graph has no triangles. O. Pretzel [28] has constructed a graph of girth six which is not a cover graph. J. Nešetřil and V. Rödl [27] have shown the existence of graphs with arbitrarily large girth which are not cover graphs.

Problem 45. *Give an explicit construction for graphs of arbitrarily large girth which are not cover graphs.*

The following problem comes from P. Erdös.

Problem 46. *Let $f(n)$ denote the minimum value of the independence number among all cover graphs on n vertices. An old construction of Erdös shows $f(n) = o(n)$. Is it true that $f(n) \geq n^{1-\varepsilon}$?*

J. Nešetřil asks the following question.

Problem 47. *If C_1 is a cover graph, does there exist a cover graph C_2 so that if the edges of C_2 are partioned into two classes, then there exists an induced copy of C_1 so that all the edges in this copy belong to the same class?*

There are a number of difficult problems involving the existence of hamiltonian cycles in graphs, in particular the well known conjecture of L. Lovasz for vertex transitive graphs. Here is a special case for ordered sets. I am not sure who first posed the problem. I first heard about it from I. Dejter who credits it to P. Erdös, but perhaps someone else should be credited. D. Kelly has posed the problem at conferences in Banff and Oberwolfach, and so has I. Havel in Prague.

Problem 48. *Consider the bipartite graph formed by the vertices in the middle two levels in the diagram for the ordered set consisting of all subsets of a set of $2n+1$ elements ordered by inclusion. Is this graph hamiltonian?*

We close this paper with a problem of Linial, Saks and Schor [25]:

Problem 49. *What is the largest integer $f(n,d)$ so that every ordered set on n points contains a d-dimensional subordered set on $f(n,d)$ points?*

These three authors have shown that there exists a constant c so that every ordered set on n points contains a subordered set on $c\sqrt{n}$ points in which all maximal chains have the same length. Of course the existence of c is easy to establish, but they have also shown that the result is best possible except for the precise value of the constant c. This problem is representative of the type of problem which deserves more attention in the future.

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References

[1] M. Aigner, "A note on merging," *Order* **2** (1985), 257–264.

[2] K. Baker, "Dimension, join independence and breadth in partially ordered sets." Unpublished notes.

[3] K. Bogart and W. Trotter, "On the complexity of posets," *Discrete Math.* **16** (1976), 71–82.

[4] K. Bogart, I. Rabinovitch and W. Trotter, "A bound on the dimension of interval orders," *J. Combin. Theory (A)* **21** (1976), 319–328.

[5] M. El–Zahar and I. Rival, "Examples of jump-critical ordered sets," *SIAM J. Alg. Discr. Meth.* (1985). To appear.

[6] M. El–Zahar and J. Schmerl, "On the size of jump-critical ordered sets," *Order* **1** (1984), 3–6.

[7] P. Fishburn, *Interval orders and interval graphs,* Wiley, New York (1985).

[8] P. Fishburn and W. Trotter, "Angle orders," *Order* **1** (1985), 333–344.

[9] M. Fredman, "How good is the information theory bound in sorting ," *Theoretical Computer Science* **1** (1979), 355–361.

[10] Z. Füredi and J. Kahn, "On the dimension of partially ordered sets," *Order*. To appear.

[11] Z. Füredi and J. Kahn. Personal Communication.

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[12] B. Ganter, P. Nevermann, K. Reuter and J. Stahl, "How small can a lattice of dimension n be?" To appear.

[13] J. Griggs, "The Sperner property," *Annals of Discrete Math.* **23** (1984), 397–408.

[14] T. Hiraguchi, "On the dimension of partially ordered sets," *Sci. Rep. Kanazawa U.* **1** (1951), 77–94.

[15] J. Kahn and M. Saks, "Balancing poset extensions," *Order* **1** (1984), 113–126.

[16] D. Kelly, "Removable pairs in dimension theory," *Order* **1** (1984), 217–218.

[17] D. Kelly, "On the dimension of partially ordered sets," *Discrete Math.* **35** (1981), 135–156.

[18] H. Kierstead, "Recursive ordered sets." To appear.

[19] H. Kierstead, "An effective version of Dilworth's theorem," *Trans. Am. Math. Soc.* **268** (1981), 63–77.

[20] H. Kierstead, G. McNulty and W. Trotter, "A Theory of recursive dimension for ordered sets," *Order* **1** (1984), 67–82.

[21] H. Kierstead, E. Szemerédi and W. Trotter, "On coloring graphs with locally small chromatic number," *Combinatorica* **4** (1984), 183–185.

[22] H. Kierstead and W. Trotter, "An extremal problem in recursive combinatorics," *Congressus Numerantium* **33**, 145–153.

[23] H. Kierstead and W. Trotter, "A Ramsey theoretic problem for ordered sets," *Discrete Math.* To appear.

[24] N. Linial, "The information theoretic bound is good for sorting," *SIAM J. Comput.* **13** (1984), 795–801.

[25] N. Linial, M. Saks and P. Schor, "Largest induced suborders satisfying the chain condition," *Order* **2** (1985), 265–268.

[26] J. Nešetřil and V. Rödl, "Combinatorial partitions of finite posets and lattices," *Algebra Universalis* **19** (1984), 106–119.

[27] J. Nešetřil and V. Rödl, "On a Probabilistic Graph-Theoretic Method," *Proc. Am. Mat. Soc.* **72** (1978), 417–421.

[28] O. Pretzel, "A Non-Covering Graph of Girth Six," *Discrete Math.* To appear.

[29] I. Rabinovitch, "An Upper Bound on the Dimension of Intervals Orders," *J. Combin. Theory (A)* (1978), 68–71.

[30] V. Rödl and W. Trotter, "An Improved bound on the Dimension of Interval Orders." To appear.

[31] M. Saks, "Balancing Extensions of Ordered Sets," *Order* **2** (1985), 327–330.

[32] J. Schmerl, "Posets with Small Width and Large Jump Number," *Order*. To appear.

[33] W. Schnyder. Personal Communication.

[34] W. Trotter, "The Dimension of the Cartesian Product of Partial Orders," *Discrete Math.* **53** (1985), 255–263.

[35] W. Trotter, "Order Preserving Embeddings of Aographs," *Springer Verlag Lecture Notes in Math.* **642** (1978), 572–579.

[36] W. Trotter and J. Moore, "The Dimension of Planar Posets," *J. Combin. Theory (B)* **22** (1977), 54–67.

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