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Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures

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Abstract

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A finite, indecomposable partially ordered set is said to be critically indecomposable if, whenever an element is removed, the resulting induced partially ordered set is not indecomposable. The same terminology can be applied to graphs, tournaments, or any other relational structure whose relations are binary and irreflexive. It will be shown in this paper that critically indecomposable partially ordered sets are rather scarce; indeed, there are none of odd order, there is exactly one of order 4, and for each even $k \geqslant 6$ there are exactly two of order k. The same applies to graphs. For tournaments, there are none have even order, there is exactly one of order 3, and for each odd $k \geqslant 5$ there are precisely three of order k. In general, for arbitrary irreflexive binary relational structures, we will see that all critical indecomposables fall into one of nine infinite classes. Four of these classes are even—they contain no structures of odd order and for even $k \geqslant 6$ they each contain (up to a certain type of equivalence) exactly one structure of order k. The five other classes are odd—they contain no structures of even order and for each odd $k \geqslant 5$ they each contain exactly one structure of order k. From this characterization of critically indecomposable structures, it will be evident that all indecomposable substructures of critically indecomposable structures are themselves critically indecomposable. Finally, it is proved that every indecomposable structure of order $n+2$ ($n \geqslant 5$) has an indecomposable substructure of order n.

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1. Basic definitions

A *binary relational structure* $\mathscr{A}$ *of type* k consists of a non-empty underlying set A together with binary relations $R_1, R_2, \ldots, R_k \subseteq A \times A$. We will write $\mathscr{A}$ as $(A; R_1, R_2, \ldots, R_k)$. The *order* of $\mathscr{A}$ is $|A|$, the number of elements of A; in particular $\mathscr{A}$ has a finite order (or, simply, $\mathscr{A}$ is *finite*) if A is finite. We will write aRb instead of $\langle a, b\rangle \in R$. If each R_i is irreflexive (that is, if aR_ia for no $a \in A$) then $\mathscr{A}$ is *irreflexive*.

The only structures that we will be encountering are finite, irreflexive, binary relational structures, so we will henceforth use the word 'structure' to refer just to these. Graphs, partially ordered sets (posets), tournaments, and oriented graphs are examples of structures of type 1. Specifically, if $\mathscr{A} = (A; R)$, then:

$\mathscr{A}$ is a *graph* iff aRb whenever bRa;

$\mathscr{A}$ is a *poset* iff aRc whenever aRb and bRc;

$\mathscr{A}$ is a *tournament* iff for distinct $a, b \in A$, wither aRb or bRa, but not both;

$\mathscr{A}$ is a *oriented graph* iff for each $a, b \in A$, not both aRb and bRa.

A structure $\mathscr{B}$ is a *substructure* of $\mathscr{A}$ if $\mathscr{A} = (A; R_1, R_2, \ldots, R_k)$, $\mathscr{B} = (B; S_1, S_2, \ldots, S_k)$, $B \subseteq A$, and $S_i = R_i \cap (B \times B)$ for each i. (In particular, 'subgraph' will always mean 'induced subgraph'.) If $\mathscr{B}$ is isomorphic to a substructure of $\mathscr{A}$, then $\mathscr{B}$ is *embeddable* in $\mathscr{A}$. We will frequently identify substructures with their underlying sets. However, for emphasis we will write $\mathscr{A} \mid B$ for the substructure of $\mathscr{A}$ having underlying set B.

Let $\mathscr{A} = (A; R_1, R_2, \ldots, R_k)$ and $\mathscr{B} = (B; S_1, S_2, \ldots, S_k)$ be structures of type k. Their *lexicographic product* $\mathscr{A} * \mathscr{B}$ (sometimes called the wreath product) is defined to be $(A \times B; T_1, T_2, \ldots, T_k)$, where $\langle a_1, b_1\rangle T_i \langle a_2, b_2\rangle$ iff either $a_iR_ia_2$ or else $a_1 = a_2$ and $b_1S_ib_2$.

A structure $\mathscr{A}$ is *indecomposable* if, whenever $\mathscr{A}$ is embeddable in $\mathscr{B}_1 * \mathscr{B}_2$, then $\mathscr{A}$ is embeddable in $\mathscr{B}_1$ or $\mathscr{B}_2$. This definition of indecomposable is not the most practicable, so we will give an equivalent and more useful one.

Let $\mathscr{A} = (A; R_1, R_2, \ldots, R_k)$ be a structure, and let $I \subseteq A$ be a subset. Then I is an *interval* if whenever $a, b \in I$ and $c \in A \setminus I$, then for each i both aR_ic iff bR_ic, and cR_ia iff cR_ib. The interval I is *nontrivial* if $2 \leq |I| < |A|$. (Sometimes in the literature intervals are referred to as *autonomous* or *partitive* sets.)

Proposition 1.1. *$\mathscr{A}$ is indecomposable iff $\mathscr{A}$ contains no nontrivial intervals.*

Proof. ($\Rightarrow$) Suppose $I \subseteq A$ is a nontrivial interval. Choose $a \in I$ and let $B = (A \setminus I) \cup \{a\}$. Clearly, $\mathscr{A}$ is not embeddable in either $\mathscr{A} \mid B$ or $\mathscr{A} \mid I$ since $|B|$, $|I| < |A|$. However, $\mathscr{A}$ is embeddable in $(\mathscr{A} \mid B) * (\mathscr{A} \mid I)$ by the function $\varphi: A \to B \times I$ where

$$\varphi(x) = \begin{cases} \langle x, a\rangle & \text{if } x \notin I, \\ \langle a, x\rangle & \text{if } x \in I. \end{cases}$$

It is easily checked that φ is an embedding.

($\Leftarrow$) Suppose $\mathscr{A}$ contains no nontrivial intervals, and let $\varphi: \mathscr{A} \to \mathscr{B}_1 * \mathscr{B}_2$ be an embedding, where $\varphi(x) = \langle \varphi_1(x), \varphi_2(x) \rangle$ for each $x \in A$. If φ_1 is one-to-one, then $\varphi_1: \mathscr{A} \to \mathscr{B}_1$ is an embedding. Therefore, we can assume φ_1 is not one-to-one, so there are distinct $x, y \in A$ such that $\varphi_1(x) = \varphi_1(y)$. Let $I = \{z \in A: \varphi_1(z) = \varphi_1(x)\}$. Clearly, I is an interval and $|I| \geq 2$. Therefore, $I = A$ since I is trivial. But then $\varphi_2: \mathscr{A} \to \mathscr{B}_2$ is an embedding. $\square$

In a structure $\mathscr{A} = (A; R_1, R_2, \ldots, R_k)$, we define the 4-ary relation $\equiv$ on A such that for any $x, y, z, w \in A$, $xy \equiv zw$ iff $x \neq y$, $z \neq w$, and for each i both of the following hold: xR_iy iff zR_iw, and yR_ix iff wR_iz. If $xy \equiv zw$ does not hold, then we write $xy \not\equiv zw$. We will refer to $(A, \equiv)$ as the *skeleton* of $\mathscr{A}$. Notice that a subset $I \subseteq A$ is an interval iff whenever $x, y \in I$ and $z \in A \setminus I$, then $xz \equiv yz$. Thus we can refer unambiguously to intervals of the skeleton $(A, \equiv)$. Also we can refer unambiguously to the indecomposability of the skeleton $(A, \equiv)$.

2. A basic theorem

In this section we state and prove a basic theorem concerning indecomposable structures. We first state another basic result, which was proved in [2] by a quite circuitous method, and then sketch a much more direct proof which is essentially the proof by Kelly (Lemma 3.5 of [1]) of the specialization of this theorem to graphs.

Theorem 2.1. *Suppose $\mathscr{A} = (A; R_1, R_2, \ldots, R_k)$ is indecomposable and $|A| \geq 3$. Then there is an indecomposable $B \subseteq A$ such that either $|B| = 3$ or $|B| = 4$.*

Sketch of Proof. Assume that $\mathscr{A}$ has no indecomposable substructure of order 3 or 4. Then there are distinct $a, b, c \in A$ such that $ab \not\equiv ac \equiv bc \not\equiv ba$. Partition $Y = A \setminus \{a, b\}$ into the following five sets (not all of which need be non-empty):

$$A = \{x \in Y: ax \equiv ac \not\equiv bx\},$$
$$B = \{x \in Y: ax \not\equiv ac \equiv bx\},$$
$$C = \{x \in Y: ax \equiv ac \equiv bx\},$$
$$D = \{x \in Y: ax \not\equiv ac \not\equiv bx \text{ and } xy \equiv ay \text{ for all } y \in C\},$$
$$E = \{x \in Y: ax \not\equiv ac \not\equiv bx \text{ and } xy \not\equiv ay \text{ for some } y \in C\}.$$

Now a straightforward, but somewhat lengthy, check by cases shows that $\{a, b\} \cup A \cup B \cup D$ is an interval, which is proper since it does not contain c. This contradicts the indecomposability of $\mathscr{A}$. $\square$

The main result of this section is the following theorem.

Theorem 2.2. *Suppose $\mathcal{A} = (A; R_1, R_2, \ldots, R_k)$ is indecomposable and $B \subseteq A$ is also indecomposable, where $3 \leq |B| \leq |A| - 2$. Then there is an indecomposable C such that $B \subseteq C \subseteq A$ and $|C| = |B| + 2$.*

Proof. We begin the proof of the theorem by proving a weaker form of it.

Claim. *Suppose $(A; R_1, R_2, \ldots, R_k)$ is indecomposable and $B \subseteq A$ is also indecomposable, where $3 \leq |B| \leq |A| - 2$. Then there are (possibly identical) $p, q \in A \setminus B$ such that $B \cup \{p, q\}$ is indecomposable.*

Suppose the claim is false. Then for each $x \in A \setminus B$, $B \cup \{x\}$ is not indecomposable. Therefore, each such $B \cup \{x\}$ has a nontrivial interval I. Then $I \cap B$ is an interval of B, and it must be trivial since B is indecomposable. Hence, for each $x \in A \setminus B$, one of the following two possibilities hold:

(2.2.1) B is an interval of $B \cup \{x\}$;

(2.2.2) there is $x' \in B$ such that $\{x, x'\}$ is an interval of $B \cup \{x\}$.

It cannot happen that both possibilities (2.2.1) and (2.2.2) hold for the same $x \in A \setminus B$, for then $B \setminus \{x'\}$ would be an interval of B, contradicting the indecomposability of B. It cannot happen that possibility (2.2.1) holds for each $x \in A \setminus B$, for then B would be an interval of A, contradicting the indecomposability of A.

If possibility (2.2.2) holds for $x \in A \setminus B$, then the corresponding x' is unique. For, suppose $x', x'' \in B$ are such that both $\{x, x'\}$ and $\{x, x''\}$ are interval of $B \cup \{x\}$. Then $\{x', x''\}$ is an interval of B, so that by the indecomposability of B it follows that $x' = x''$.

Let $X = \{x \in A \setminus B : \{x, x'\}$ is an interval of $B \cup \{x\}$ for some $x' \in B\}$. By the preceding discussion $X \neq \emptyset$, so consider some $d \in X$. Since A is indecomposable, $\{d, d'\}$ is not an interval of A. Thus, there is $c \in A \setminus \{d, d'\}$ such that $cd \not\equiv cd'$. Because $\{d, d'\}$ is an interval of $B \cup \{d\}$, it must be that $c \in B$.

In fact, $c \in X$. To see this, suppose $c \notin X$ so that by possibility (2.2.1), B is an interval of $B \cup \{c\}$. Since, by hypothesis, $B \cup \{c, d\}$ is not indecomposable, there is an interval J of $B \cup \{c, d\}$. Clearly, $J \neq \{c, d\}$; for then, if $y \in B \setminus \{d'\}$, then $yd' \equiv yd \equiv yc \equiv d'c$, so that $B \setminus \{d'\}$ would be an interval of B. Also, $J \neq B \cup \{d\}$ since $cd \not\equiv cd'$; and $J \neq B \cup \{c\}$, for then B would be an interval of $B \cup \{d\}$. We have just shown that $J \cap B \neq \emptyset$ and $J \cap B \neq B$; but $J \cap B$ is an interval of B and B is indecomposable, so that $|J \cap B| = 1$. Moreover, $J \cap B = \{d'\}$. For if $J \cap B = \{y\}$ with $y \neq d'$, then $\{d, y\}$ would be an interval of $B \cup \{d\}$ different from $\{d, d'\}$, contradicing the uniquness of d' in (2.2.2). Therefore, we have reduced J to being one of $\{d, d'\}$, $\{c, d'\}$ or $\{c, d, d'\}$. The first is impossible since $cd \not\equiv cf'$. For the second and third alternatives, choose $b \in B \setminus \{d'\}$. Then $bd' \equiv bc \equiv dc'$, so that $B \setminus \{d'\}$ is an interval of B, contradicting the indecomposability of B. This proves $c \in X$.

Next we show that $c' = d'$. To see this suppose to the contrary that $c' \neq d'$. By hypothesis, $B \cup \{c, d\}$ is not indecomposable, so $B \cup \{c, d\}$ has a nontrivial interval Y. It is not the case that $Y = \{c, d\}$, for if that were so then for any $a \in B \setminus \{c', d\}$, $ad' \equiv ad \equiv ac \equiv ac'$, implying that $\{c', d'\}$ is an interval of B. Thus $Y \cap B \neq \emptyset$, so either $|Y \cap B| = 1$ or $Y \cap B = B$. Consider first that $Y \cap B = \{y\}$. If $c \neq Y$, then by the uniqueness of d' it follows that $y = d'$; but $cd \not\equiv cd'$ yields a contradication. If $d \notin Y$, then by the uniqueness of c', it follows that $y = c'$; but then $dc \equiv dc' \equiv d'c' \equiv d'c$ yield a contradiction. Thus $\{c, d\} \subseteq Y$, so that $\{c', d'\} \subseteq Y$, implying the contradiction that $c' = d'$. Therefore, we have that $|Y \cap B| \neq 1$, so that $B \subseteq Y$.

Now, if $d \in Y$, then $cd \not\equiv cd'$ implies $c \in Y$, so Y is a trivial interval of $B \cup \{c, d\}$. Thus $d \notin Y$; but then $dc \equiv dc' \equiv d'c' \equiv d'c$ gives a contradiction unless $c \notin Y$, so $Y = B$. However, $Y = B$ implies that $B \setminus \{c'\}$ is an interval of B. For consider arbitrary $a \in B \setminus \{c'\}$; then $ac' \equiv ac \equiv c'c$. This again contradicts the indecomposability of B, thus shown that $c' = d'$.

For the final contradiction, just notice that $\{z \in X' z' = d'\}$ is a nontrivial interval of A. This completes the proof of the claim. □

We now deduce Theorem 2.2 from the claim. Suppose that Theorem 2.2 is false and $(A; R_1, R_2, \ldots, R_k)$ and B form a counterexample in which $|A|$ is minimal. The claim then implies that $|A| = |B| + 3$, and that $B \cup \{q\}$ is indecomposable for some $q \in A \setminus B$. Let $X = \{x \in A \setminus (B \cup \{q\}) : \{x, x'\}$ is an interval of $B \cup \{x, q\}$ for some $x' \in B \cup \{q\}\}$. As in the proof of the claim, $X \neq \emptyset$ and for each $x \in X$ there is a unique $x' \in B \cup \{q\}$. Select some $d \in X$. Since A is indecomposable, $\{d, d'\}$ is not an interval of A, so there exists $c \in A \setminus \{d, d'\}$ such that $cd \not\equiv cd'$. Clearly $c \notin B \cup \{q\}$, because $\{d, d'\}$ is an interval of $B \cup \{q\}$. Therefore $\{c\} = A \setminus (B \cup \{d, q\})$.

Suppose $c \in X$. Then as in the proof of the claim, $c' = d'$. Thus, $\{c, d, d'\}$ is an interval of A, contradicting its decomposability.

So suppose $c \notin X$. Then as in the proof of the claim, $B \cup \{q\}$ is an interval of $B \cup \{q, c\}$. If $d' \neq q$, then $\{d, d'\}$ is an interval of $B \cup \{d\}$, and B is an interval of $B \cup \{c\}$. It then follows, as in the proof of the claim, that $B \cup \{c, d\}$ is indecomposable.

Thus, we can suppose, in addition to $c \notin X$, that $d' = q$. Since $B \cup \{d, c\}$ is not indecomposable, it has a nontrivial interval Y. Now $Y \cap B$ is an interval of B; but B being indecomposable implies $Y \cap B$ is a trivial interval of B. Thus, $B \subseteq Y$, $Y = \{d, c\}$, $Y = \{d, c, y\}$, $Y = \{d, y\}$ or $Y = \{c, y\}$ for some $y \in B$. In any case a contradiction will ensue:

if $B \subseteq Y$ and $d \notin Y$, then B is an interval of $B \cup \{q\}$;
if $B \subseteq Y$ and $d \in Y$, then $cd \equiv cq$, contradicting the definition of c;
if $Y = \{d, c\}$, then $\{d, q, c\}$ is an interval of A;
if $Y = \{d, c, y\}$, then $\{d, q, c, y\}$ is an interval of A;
if $Y = \{d, y\}$, then $\{d, y, q\}$ is an interval of A;
if $Y = \{c, y\}$, then $cd \equiv yd \equiv yd' \equiv cd'$, contradicting $cd \not\equiv cd'$. □

Corollary 2.3. *Suppose $(A; R_1, R_2, \dots, R_k)$ is indecomposable and $|A| \geq 3$. Then there is an indecomposable $B \subseteq A$ such that $|B| = |A| - 1$ or $|B| = |A| - 2$.*

It will be shown later, in Theorem 5.9, that Corollary 2.3 can be improved in case that $|A| \geq 7$ so as to be able to conclude that always $|B| = |A| - 2$.

3. Critical indecomposability

Corollary 2.3 suggests a definition which will be central to the rest of this paper. An indecomposable structure $\mathscr{A} = (A; R_1, R_2, \dots, R_k)$ is *critically indecomposable* if whenever $a \in A$, then $A \setminus \{a\}$ is not indecomposable. Notice that we can refer unambiguously to a skeleton $(A, \equiv)$ being critically indecomposable.

Theorems 2.1 and 2.2 imply that more is true of critically indecomposable structures than is immediately apparent.

Corollary 3.1. *Suppose that $\mathscr{A}$ is a critically indecomposable structure of order n and that $3 \leq m \leq n$. Then $\mathscr{A}$ has an indecomposable substructure of order m iff $n - m$ is even.*

Some examples of critically indecomposable structures are given in Section 4. In Section 5 it is shown that this list of examples is complete, up to isomorphism of skeletons.

4. Examples of critically indecomposable structures

In this section we will give some examples of critically indecomposable graphs, posets, tournaments and linear directed graphs, and also of some critically indecomposabie structures of type 2.

First we consider graphs. For each $r \geq 2$ we will define a graph $\mathscr{G}_r = (V_r; E_r)$ of order $2r$. Let $V_r = \{a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r\}$, where $\{a_1, a_2, \dots, a_r\}$ and $\{b_1, b_2, \dots, b_r\}$ are independent sets and a_i and b_j are adjacent iff $i \geq j$. Let $\mathscr{G}'_r = (V_r; E'_r)$ be the complementary graph of $\mathscr{G}_r$. Note that $\mathscr{G}_2 \cong \mathscr{G}'_2$, but $\mathscr{G}_r \ncong \mathscr{G}'_r$ for $r > 2$.

Lemma 4.1. *For each $r \geq 2$ the graphs $\mathscr{G}_r$ and $\mathscr{G}'_r$ are indecomposable.*

Proof. Clearly, $\mathscr{G}_r$ is critically indecomposable iff $\mathscr{G}'_r$ is. We will show that $\mathscr{G}_r$ is indecomposable. This is done by induction or r. By inspection, $\mathscr{G}_2$ is indecomposable. Now suppose that I is a nontrivial interval of $\mathscr{G}_{r+1}$, where $r \geq 2$. Since

$a_i \leftrightarrow b_{r+2-i}$ is an automorphism of $\mathscr{G}_{r+1}$, we can, without loss of generality assume that $a_i \notin I$ for some i. Now, let i be the smallest such i. Since $V_{r+1} \setminus \{a_i, b_i\}$ is isomorphic to $\mathscr{G}_r$ and $I \setminus \{b_i\}$ is an interval of $V_{r+1} \setminus \{a_i, b_i\}$, it follows from the inductive hypothesis that $I \setminus \{b_i\}$ is trivial, so that either $I = \{x, b_i\}$ for some $x \in V_{r+1} \setminus \{a_i, b_i\}$, $I = V_{r+1} \setminus \{a_i, b_i\}$, or $I = V_{r+1} \setminus \{a_i\}$. The first alternative is impossible, for if $x = b_j \neq b_i$, then $a_i b_i \not\equiv a_i b_j$ since $1 = i < j$, and if $x = a_j$, then $j \neq i$ and $b_i a_j \not\equiv b_j b_i$. The second alternative is impossible, for if $i \leq r$ then $b_i a_{i+1} \not\equiv b_i b_{i+1}$, and if $i = r+1$ then $a_{r+1} a_1 \not\equiv a_{r+1} b_1$. Finally, the third alternative is also impossible, for $a_i b_i \not\equiv a_i a_j$ where $j \neq i$. □

Each of the graphs $\mathscr{G}_r$ and $\mathscr{G}'_r$ is a comparability graph, as can be seen by considering the posets $\mathscr{P}_r = (V_r; P_r)$ and $\mathscr{P}'_r = (V_r; P'_r)$, where

$$xP_r y \Leftrightarrow x = a_i \text{ and } y = b_j \text{ for some } i \geq j,$$

and

$$xP'_r y \Leftrightarrow \langle x, y \rangle \in \{\langle a_i, b_j \rangle, \langle a_i, a_j \rangle, \langle b_i, b_j \rangle\} \text{ for some } i < j.$$

Clearly $\mathscr{P}_r \cong \mathscr{P}'_r$ iff $r = 2$.

For $r \geq 2$, let $\mathscr{B}_r$ be the structure $(V_r; P_r, P'_r)$.

Proposition 4.2. *For each $r \geq 2$, $\mathscr{B}_r$ is critically indecomposable.*

Proof. The indecomposability of $\mathscr{B}_r$ follows from the indecomposability of $\mathscr{G}_r$, shown in Lemma 4.1. To show that $\mathscr{B}_r$ is critically indecomposable, consider the substructure $V_r \setminus \{x\}$, for arbitrary $x \in V_r$. Depending upon the choice of x, we can find a nontrivial interval I of $V_r \setminus \{x\}$: if $x = a_r$, then $I = V_r \setminus \{a_r, b_r\}$; if $x = b_1$, then $I = V_r \setminus \{a_1, b_1\}$; if $x = a_i$ for $i < r$, then $I = \{b_i, b_{i+1}\}$; and if $x = b_i$ for $1 < i$, then $I = \{a_{i-1}, a_i\}$. □

Proposition 4.3. *For each $r \geq 2$ the graphs $\mathscr{G}_r$ and $\mathscr{G}'_r$ and the posets $\mathscr{P}_r$ and $\mathscr{P}'_r$ are critically indecomposable.*

Proof. The indecomposability of $\mathscr{P}_r$ and $\mathscr{P}'_r$ follows from the indecomposability of their comparability graphs $\mathscr{G}_r$ and $\mathscr{G}'_r$, which was shown in Lemma 4.1. Then the critical indecomposability of these structures follows from the critical indecomposability of $\mathscr{B}_r$ which was shown in Proposition 4.2. □

There are still five more infinite families of critically indecomposable structures which will be presented in this section. The proofs that these structures are indeed critically indecomposable are quite easy and much like the previous proofs. We will leave these proofs to the enterprising reader.

Three of these families consist of tournaments. For $i = 1, 2, 3$ and $r \geq 2$ we will define tournaments $\mathscr{T}_r^{(i)}$ of order $2r+1$, where $\mathscr{T}_r^{(i)} = (T_r^{(i)}; \xrightarrow{(i)})$.

Let $T_r^{(1)} = \{c_0, c_1, \ldots, c_{2r}\}$, and let $c_i \xrightarrow{(1)} c_j$ iff $i + k \equiv j \pmod{2r+1}$ for some $k = 1, 2, \ldots, r$.

Let $T_r^{(2)} = \{a_0, a_1, \ldots, a_r, \; b_1, b_2, \ldots, b_r\}$ and let $x \xrightarrow{(2)} y$ iff one of the following holds:

$x = b_i$, $y = b_j$ and $i < j$;
$x = b_i$, $y = a_j$ and $i \leq j$;
$x = a_i$, $y = b_j$ and $i < j$;
$x = a_j$, $y = a_i$ and $i < j$.

Let $T_r^{(3)} = \{b, a_1, a_2, \ldots, a_{2r}\}$. Let $x \xrightarrow{(3)} y$ iff one of the following holds:

$x = a_i$, $y = a_j$ and $i < j$;
$x = b$ and $y = a_i$ for odd i;
$x = a_i$ and $y = b$ for even i.

Notice that the tournaments $\mathcal{T}_r^{(1)}$, $\mathcal{T}_r^{(2)}$ and $\mathcal{T}_r^{(3)}$ are pairwise non-isomorphic. Each is self-dual.

Proposition 4.4. *For each $r \geq 2$ the tournaments $\mathcal{T}_r^{(1)}$, $\mathcal{T}_r^{(2)}$ and $\mathcal{T}_r^{(3)}$ are critically indecomposable.* $\square$

Tournaments are special kinds of oriented graphs. We next present infinite family of critically indecomposable oriented graphs. For $r \geq 2$ let $\mathcal{D}_r = (T_r^{(2)}; F_r)$ be the oriented graph whose underlying set is the same as the underlying set of $\mathcal{T}_r^{(2)}$, and where xF_ry iff $x \xrightarrow{(2)} y$ and $\{x, y\} \cap \{b_1, b_2, \ldots, b_r\} \neq \emptyset$.

Proposition 4.5. *For each $r \geq 2$, $\mathcal{D}_r$ is critically indecomposable.*

Let $\mathcal{D}_r' = (T_r^{(2)}; F_r, F_r')$ be the structure of type 2, where $xF_r'y$ iff $x \xrightarrow{(2)} y$ but not xF_ry.

Proposition 4.6. *For each $r \geq 2$, $\mathcal{D}_r'$ is critically indecomposable.*

5. The characterization of critically indecomposable skeletons

In Section 4 some examples of critically indecomposable structures were presented. We show in this section that, up to isomorphism of skeletons, this list is complete.

Theorem 5.1. *Suppose that $(A, \equiv)$ is a critically indecomposable skeleton. Then for some $r \geq 2$, $(A, \equiv)$ is isomorphic to the skeleton of one of the following structures: $\mathcal{G}_r$, $\mathcal{P}_r$, $\mathcal{P}_r'$, $\mathcal{B}_r$, $\mathcal{T}_r^{(1)}$, $\mathcal{T}_r^{(2)}$, $\mathcal{T}_r^{(3)}$, $\mathcal{D}_r$, $\mathcal{D}_r'$.*

We remark that no two structures in this list of critically indecomposable structures have isomorphic skeletons (with one exception: $\mathcal{P}_2 \cong \mathcal{P}_2'$).

The proof of Theorem 5.1 will include a series of lemmas. Lemmas 5.2, 5.3, and 5.5 are, in essence, trivial as they just verify Theorem 5.1 in case $|A| \leq 5$. Lemmas 5.4 and 5.6 are the keys to the proof of Theorem 5.1, giving the structural properties needed in order to do an inductive argument.

Lemma 5.2. *Suppose* $(A, \equiv)$ *is a skeleton of order* 3, *where* $A = \{a, b, c\}$. *Then* $(A, \equiv)$ *is indecomposable iff* $ab \not\equiv ac \not\equiv bc \not\equiv ba$.

Proof. Obvious. □

Lemma 5.3. *Suppose* $(A, \equiv)$ *is a critically indecomposable skeleton of order* 4. *Then* $(A, \equiv)$ *is isomorphic to the skeleton of* $\mathcal{G}_2$, $\mathcal{P}_2$ *or* $\mathcal{B}_2$.

Proof. Suppose $(A, \equiv)$ is critically indecomposable with 4 elements a, b, c, d. Since each subskeleton of order 3 is not indecomposable, we can apply Lemma 5.2 to it, and we will do so many times without specific reference. Considering $\{a, b, c\}$, we can without loss of generality assume $ba \equiv bc$. Now consider the subskeleton $\{b, c, d\}$. Notice that $bc \not\equiv bd$, as otherwise $\{a, c, d\}$ would be an interval. Hence, it must be that either $cb \equiv cd$ or $db \equiv dc$.

First suppose $cb \equiv cd$. Now consider the subskeleton $\{a, b, d\}$. If $ab \equiv ad$, then $\{b, d\}$ is an interval of $(A, \equiv)$; if $ba \equiv bd$, then $\{a, c, d\}$ is an interval of $(A, \equiv)$; therefore, it follows that $da \equiv db$. Similarly, by considering the subskeleton $\{d, c, a\}$, we get that $ac \equiv ad$. Thus,

$$ab \equiv cb \equiv cd$$

and

$$bd \equiv ad \equiv ac.$$

Also, $ab \neq bd$ (as otherwise $\{b, c, d\}$ would be an interval of $(A, \equiv)$) and $ab \not\equiv db$ (as otherwise $\{a, c, d\}$ would be an interval of $(A, \equiv)$). This leaves as unsettled only whether or not $ab \equiv ba$ and whether or not $bd \equiv db$. If $ab \not\equiv ba$ and $bd \not\equiv db$, then $(A, \equiv)$ is isomorphic to the skeleton of $\mathcal{B}_2$; if $ab \equiv ba$ and $bd \not\equiv db$ (or $ab \not\equiv ba$ and $bd \equiv db$), then $(A, \equiv)$ is isomorphic to the skeleton of $\mathcal{P}_2$; and if $ab \equiv ba$ and $bd \equiv db$, then $(A, \equiv)$ is isomorphic to the skeleton of $\mathcal{G}_2$.

Next, suppose $db \equiv dc$ rather than $cb \equiv cd$. Consider the subskeleton $\{a, b, d\}$. If $ba \equiv bd$, then $\{a, c, d\}$ is an interval of $(A, \equiv)$; if $da \equiv db$, then $\{a, b, c\}$ is an interval of $(A, \equiv)$; therefore, it follows that $ab \equiv ad$. But then, with relabelling the points, we are in the situation of the previous paragraph. □

Lemma 5.4. *Suppose* $(A, \equiv)$ *is a critically indecomposable skeleton. Then there exist distinct* $a, b, a', b' \in A$ *such that* (1), (2) *and* (3) *hold.*

(1) $A_0 = A \setminus \{a', b'\}$ *is indecomposable.*

(2) $\{a, a'\}$ *is an interval of* $A \setminus \{b'\}$.

(3) $\{b, b'\}$ *is an interval of* $A \setminus \{a'\}$.

Proof. By Corollary 2.3 there are distinct $x, y \in A$ such that $A\setminus\{x, y\}$ is indecomposable. Since $(A, \equiv)$ is critically indecomposable, $A\setminus\{x\}$ and $A\setminus\{y\}$ are not indecomposable, so they have nontrivial intervals I and J respectively. Then $I\setminus\{y\}$ and $J\setminus\{x\}$ are intervals of $A\setminus\{x, y\}$, so they must be trivial. If $I\setminus\{y\} = J\setminus\{x\} = A\setminus\{x, y\}$, then $A\setminus\{x, y\}$ is a nontrivial interval of $(A, \equiv)$. Without loss of generality, $|I\setminus\{y\}| = 1$, so let $I = \{y, y'\}$.

Suppose now that $J = \{x, x'\}$. If $x' = y'$, then $\{x, y, x'\}$ is a nontrivial interval of $(A, \equiv)$, contradicting the indecomposability of $(A, \equiv)$. If $x' \neq y'$, then let $a = x'$, $b = y'$, $a' = x$ and $b' = y$.

Suppose next that $J = A\setminus\{x, y\}$. Now $A\setminus\{y'\}$ is not indecomposable, so it must contain an interval J'. Furthermore, since $A\setminus\{x, y'\}$ is isomorphic to $A\setminus\{x, y\}$, it is indecomposable, so that $J' \cap (A\setminus\{x, y'\})$ must be a trivial interval of $A\setminus\{x, y\}$. Thus, either $J' = A\setminus\{x, y'\}$ or else $J' = \{x, x''\}$ for some $x'' \in A\setminus\{x, y'\}$. The first alternative implies that $A\setminus\{x\}$ is an interval of $(A, \equiv)$. Therefore, $J' = \{x, x''\}$. Now $x'' \neq y$ as otherwise $\{x, y, y'\}$ would be an interval of A. Thus, x'', y, x, y'' are distinct, so let $a = x''$, $b = y$, $a' = x$ and $b' = y'$. Clearly, (1), (2) and (3) hold. $\square$

Lemma 5.5. *Suppose $(A, \equiv)$ is a critically indecomposable skeleton of order 5. Then $(A, \equiv)$ is isomorphic to the skeleton of $\mathcal{T}_2^{(i)}$ $(i = 1, 2, 3)$, $\mathcal{D}_2$ or $\mathcal{D}_2'$.*

Proof. According to Lemma 5.4, we can get $A = \{a, b, c, a', b'\}$ where $\{a, b, c\}$ is indecomposable and

(5.5.1) $a'b \equiv ab \equiv ab', \qquad a'c \equiv ac, \qquad b'c \equiv bc.$

Since $\{a, b, c\}$ is indecomposable, we have from Lemma 5.2 that

(5.5.2) $ab \not\equiv ac \not\equiv bc \not\equiv ba$

Consider $\{b, c, a', b'\}$, which, not being indecomposable, must contain a nontrivial interval. There are six 2-element subsets and four 3-element subsets for a total of ten possible nontrivial intervals. Seven of these can be immediately excluded, for if any of them were an interval, then we obtain a contradiction to (5.5.2) as follows: If one of the intervals of $\{b, c, a', b'\}$ is:

$\{b, c\}$, then $ac \equiv a'c \equiv a'b \equiv ab$;

$\{b, a'\}$, then $cb \equiv ca' \equiv ca$;

$\{c, a'\}$, then $bc \equiv ba' \equiv ba$;

$\{a', b'\}$, then $cb \equiv cb' \equiv ca' \equiv ca$;

$\{c, a', b'\}$, then $bc \equiv ba' \equiv ba$;

$\{b, b, b'\}$, then $ac \equiv a'c \equiv a'b \equiv ab$;

$\{b, a', b'\}$, then $ca \equiv ca' \equiv cb' \equiv cb$.

Also $\{b, b'\}$ cannot be an interval of $\{b, c, a', b'\}$, for then it would be an interval of $\{a, b, c, a', b'\}$.

The only possibilities for nontrivial intervals of $\{b, c, a', b'\}$ are $\{c, b'\}$ and $\{b, c, a'\}$. Similarly, the only possibilites for nontrivial intervals of $\{a, c, a', b'\}$

are $\{c, a'\}$ and $\{a, c', b'\}$. Taking into account symmetry, there are only three cases to consider.

Case 1: *$\{c, b'\}$ is an interval of $\{b, c, a', b'\}$, and $\{c, a'\}$ is an interval of $\{a, c, a', b'\}$.*

This yields

$$a'c \equiv a'b' \equiv cb' \equiv cb \equiv b'b \equiv ac \equiv aa'.$$

From (5.5.1) and (5.5.2) we also have

$$a'b \equiv ab \equiv ab' \not\equiv cb.$$

If $ba' \not\equiv a'c$, then $\{a, a', b, b'\}$ would be indecomposable, so $ba' \equiv a'c$. Thus we get a skeleton isomorphic to the skeleton of $\mathcal{T}_2^{(1)}$, the isomorphism being $\langle a, b, b', c, a' \rangle \mapsto \langle c_0, c_1, c_2, c_3, c_4 \rangle$.

Case 2: *$\{c, b'\}$ is an interval of $\{b, c, a', b'\}$ and $\{a, c, b'\}$ is an interval of $\{a, c, a', b'\}$.*

This yields

$$ca \equiv ca' \equiv aa' \equiv b'a', \qquad ab \equiv a'b \equiv ab', \qquad bc \equiv b'c \equiv bb'.$$

One can check that in order that $\{a, a', b, b'\}$ not be indecomposable, either

$$ab \equiv a'a, \qquad ab \equiv b'b, \quad \text{or} \quad ab \equiv aa'.$$

If $ab \equiv a'a$, then $ab \equiv ac$, which contradicts (5.5.2). If $ab \equiv b'b$, then $ab \equiv b'b \equiv cb$, which also is a contradiction. Therefore $ab \equiv aa'$, so that

$$ca \equiv ca' \equiv aa' \equiv b'a' \equiv ab \equiv a'b \equiv ab'.$$

There are now three possibilities;

$$bc \equiv ab, \qquad bc \equiv cb, \qquad cb \not\equiv bc \not\equiv ab.$$

The first gives a skeleton isomorphic to the skeleton of $\mathcal{T}_2^{(2)}$, the second to $\mathcal{D}_2$, and the third to $\mathcal{D}_2'$.

Case 3: *$\{b, c, a'\}$ is an interval of $\{b, c, a', b'\}$, and $\{a, c, b'\}$ is an interval of $\{a, c, a', b'\}$.*

This yields

$$b'b \equiv b'c \equiv b'a' \equiv aa' \equiv ca' \equiv ca \equiv bc \quad \text{and} \quad a'b \equiv ab \equiv ab'.$$

One can check that in order that $\{a, b, a', b'\}$ not be indecomposable, either $ab \equiv aa'$ or $ab \equiv a'a$. But $ab \equiv a'a$ implies $ab \equiv ac$, contradicting (5.5.2), so that $ab \equiv aa'$ and $ab \not\equiv a'a$. This results in a skeleton which is isomorphic to the skeleton of $\mathcal{T}_2^{(3)}$, the isomorphism being $\langle c, a, b', b \rangle \mapsto \langle b, a_1, a_2, a_3, a_4 \rangle$. □

Lemma 5.6. *Suppose $(A, \equiv)$ and $a, b, a', b' \in A$ are as in Lemma 5.4, and that $|A| \geqslant 6$. Then* (3′) *holds and either* (1′) *or* (2′) *holds.*

(1′) *There is $p \in A_0 \setminus \{a, b\}$ such that $\{a', p\}$ is an interval of $A \setminus \{b\}$.*

(2′) *$A \setminus \{a', b\}$ is an interval of $A \setminus \{b\}$.*

(3′) *There is $q \in A_0 \setminus \{a, b\}$ such that $\{b', q\}$ is an interval of $A \setminus \{a\}$.*

Proof. Let $a, b, a', b' \in A$ be as in Lemma 5.4. Since $A \setminus \{a\}$ is not indecomposable, there exists a nontrivial interval $I \subseteq A \setminus \{a\}'$ similarly, there is a nontrivial interval $J \subseteq A \setminus \{b\}$. Each of $I \setminus \{b'\}$ and $J \setminus \{a'\}$ is an interval of $A \setminus \{a, b'\}$ and $A \setminus \{b, a'\}$ respectively, and there intervals are trivial since both $A \setminus \{a, b'\}$ and $A \setminus \{b, a'\}$ are indecomposable, as they are isomorphic to A_0. If $I = A \setminus \{a, b'\}$ $J = A \setminus \{b, a'\}$, then $I \cap J = A \setminus \{a, b, a', b'\}$ would be a nontrivial interval of A_0 (since $|A| > 5$). Therefore, without loss of generality, we can assume $I = \{q, b'\}$ for some $q \in A \setminus \{a, b'\}$. Moreover, $q \neq a'$ as otherwise $I = \{a', b'\}$ would be nontrivial interval of A_0; and also $q \neq b$ as otherwise $I = \{b, b'\}$ would be a nontrivial interval of A. Therefore, $q \in A_0 \setminus \{a, b\}$ and (3′) holds.

Similarly, we see that either (2′) $J = A \setminus \{a', b\}$ or (1′) $J = \{a', p\}$ for some $p \in A_0 \setminus \{a, b\}$. □

We now make some ad hoc definitions. Let $(A_0, \equiv)$ be a critically indecomposable skeleton. Then $\langle a, b \rangle$ is a *building pair* if $a, b \in A_0$ are distinct and, of the following three conditions, (3) holds and either (1) or (2) holds:

(1) There is $p \in A_0 \setminus \{a, b\}$ such that $\{a, p\}$ is an interval of $A_0 \setminus \{b\}$.

(2) $A_0 \setminus \{a, b\}$ is an interval of $A_0 \setminus \{b\}$.

(3) There is $q \in A_0 \setminus \{a, b\}$ such that $\{b, q\}$ is an interval of $A_0 \setminus \{a\}$.

The building pair $\langle a, b \rangle$ is of the first or second kind according to whether condition (1) or (2) holds. If $\langle a, b \rangle$ is a building pair and q is as in (3), then we say that q is a *mate* for b; in addition, if $\langle a, b \rangle$ is of the first kind, then we also say that p is a *mate* for a.

Remark 5.7. Suppose a, b, A_0, A are as in Lemma 5.6. If $|A_0| \geq 4$, then A_0 is critically indecomposable by Theorem 2.2. It is easily see that $\langle a, b \rangle$ is a building pair for A_0 and that the element q in (3′) is a mate for b. In addition, if (2′) holds, then $\langle a, b \rangle$ is of the second kind, and if (1′) holds then $\langle a, b \rangle$ is of the first kind and the element p in (1′) is a mate for a.

It is quite easy to determine by inspection all the buildings pairs in each of the standard examples of critically indecomposable structures that were presented in Section 4. In all cases, no building pair is both of the first and second kind. Also in all cases, all mates are unique, so we will refer to 'the' mate. If $\langle a, b \rangle$ is a building pair of the first kind, then we will not distinguish between the building pairs $\langle a, b \rangle$ and $\langle b, a \rangle$.

The only building pairs of the first kind for $\mathcal{G}_r$ $(r \geq 2)$ are $\langle a_i, b_{i+1} \rangle$ for $i = 1, 2, \ldots, r-1$ and $\langle a_i, b_i \rangle$ for $i = 2, 3, \ldots, r-1$. In the first case a_{i+1} and b_i are the mates of a_i and b_{i+1} respectively; in the second case a_{i-1} and b_{i+1} are the mates of a_i and b_i respectively. The only building pairs of the second for $\mathcal{G}_r$ $(r \geq 3)$ are $\langle a_1, b_1 \rangle$ and $\langle b_r, a_r \rangle$, in which the mate of b_1 and b_2 and the mate of a_r is a_{r-1}. The only building pairs of the second kind for $\mathcal{G}_2$ are $\langle a_1, b_1 \rangle$, $\langle b_2, a_2 \rangle$, $\langle a_2, a_1 \rangle$ and $\langle b_1, b_2 \rangle$, with mates b_2, a_1, b_1 and a_2 respectively.

The building pairs for $\mathscr{P}_r$, $\mathscr{P}'_r$ and $\mathscr{B}_r$ (for $r \geq 2$), their kinds and the mates involved are identical to those of $\mathscr{G}_r$.

All the building pairs of $\mathscr{T}^{(1)}_r$ $(r \geq 2)$ are of the first kind, and they are $\langle c_i, c_{i+r} \rangle$ for $i = 0, 1, \ldots, 2r$, where $i + r$ is taken modulo $2r + 1$. The mates of c_i and c_{i+r} are c_{i-1} and c_{i+r+1} respectively.

The only building pairs of the first kind for $\mathscr{T}^{(2)}_r$ are $\langle a_i, b_i \rangle$ for $i = 1, 2, \ldots, r-1$ and $\langle a_i, b_{i+1} \rangle$ for $i = 1, 2, \ldots, r-1$. In the first case the mates of a_i and b_i are a_{i-1} and b_{i+1} respectively; and in the second case the mates of a_i and b_{i+1} are a_{i+1} and b_i respectively. The only building pairs of the second kind for $\mathscr{T}^{(2)}_r$ are $\langle b_r, a_r \rangle$ and $\langle b_1, a_0 \rangle$, in which the mate of a_r is a_{r-1} and the mate of a_0 is a_1.

The building pairs for $\mathscr{D}_r$ and $\mathscr{D}'_r$ $(r \geq 2)$, their kinds and mates involved are identical to those of $\mathscr{T}^{(2)}_r$.

The only building pairs of the first kind for $\mathscr{T}^{(3)}_r$ $(r \geq 2)$ are $\langle a_i, a_{i+1} \rangle$ for $i = 2, 3, \ldots, 2r-2$; the mate for a_i is a_{i+2} and the mate for a_{i+1} is a_{i-1}. The only building pairs of the second kind for $\mathscr{T}^{(3)}_r$ are $\langle a_1, a_2 \rangle$ and $\langle a_{2r-1}, a_{2r} \rangle$, in which the mates are a_3 and a_{2r-2} respectively.

It now becomes an easy matter to complete the proof of Theorem 5.1 by induction on the order of the critically indecomposable skeletons. Suppose $(A, \equiv)$ is a critically indecomposable skeleton of order k, where $k > 5$. Let $a, b, a', b' \in A$ be as in Lemma 5.6, letting $A_0 = A \setminus \{a', b'\}$. Then $(A_0, \equiv)$ is indecomposable, so by Theorem 2.2, $(A_0, \equiv)$ is critically indecomposable, so it is isomorphic to the skeleton of some $\mathscr{G}_r$, $\mathscr{P}_r$, $\mathscr{P}'_r$, $\mathscr{B}_r$, $\mathscr{T}^{(1)}_r$, $\mathscr{T}^{(2)}_r$, $\mathscr{T}^{(3)}_r$, $\mathscr{D}_r$ or $\mathscr{D}'_r$, and by Remark 5.7 $\langle a, b \rangle$ is a building pair of $(A_0, \equiv)$. We have seen exactly which are the building pairs of each of these structures, their kinds and the mates involved. If $\langle a, b \rangle$ is of the first kind, then the mates of a and b are the elements p and q from 5.6(1′) and 5.6(3′). It is easily seen that $aa' \equiv ap$, $a'b' \equiv pb$ and $bb' \equiv bq$. Then checking each of the various cases (a task left to the reader) we see that $(A, \equiv)$ is isomorphic to the skeleton of $\mathscr{G}_{r+1}$, $\mathscr{P}_{r+1}$, $\mathscr{P}'_{r+1}$, $\mathscr{B}_{r+1}$, $\mathscr{T}^{(1)}_{r+1}$, $\mathscr{T}^{(2)}_{r+1}$, $\mathscr{T}^{(3)}_{r+1}$, $\mathscr{D}_{r+1}$ or $\mathscr{D}'_{r+1}$ respectively. If $\langle a, b \rangle$ is of the second kind, then q still is the mate of b, and, as is easily seen, $bb' \equiv bq$ and $a'a \equiv aq \equiv a'b'$. Again, in each of the various cases we get that $(A, \equiv)$ is isomorphic to the skeleton of $\mathscr{G}_{r+1}$, $\mathscr{P}_{r+1}$, $\mathscr{P}'_{r+1}$, $\mathscr{B}_{r+1}$, $\mathscr{T}^{(1)}_{r+1}$, $\mathscr{T}^{(2)}_{r+1}$, $\mathscr{T}^{(3)}_{r+1}$, $\mathscr{D}_{r+1}$ or $\mathscr{D}'_{r+1}$ respectively. (There is one small exception to the previous discussion: if $(A_0, \equiv)$ is isomorphic to the skeleton of $\mathscr{P}_2$, then $(A, \equiv)$ is isomorphic to the skeleton of either $\mathscr{P}_3$ or $\mathscr{P}'_3$.)

This completes the proof of Theorem 5.1. $\square$

Corollary 5.8. (1) *$\mathscr{G}$ is a critically indecomposable graph iff $\mathscr{G}$ is isomorphic to either $\mathscr{G}_r$ or $\mathscr{G}'_r$ for $r \geq 2$.*

(2) *$\mathscr{P}$ is a critically indecomposable poset iff $\mathscr{P}$ is isomorphic to either $\mathscr{P}_r$ or $\mathscr{P}'_r$ for $r \geq 2$.*

(3) *$\mathscr{T}$ is a critically indecomposable tournament iff $\mathscr{T}$ is isomorphic to $\mathscr{T}^{(1)}_r$, $\mathscr{T}^{(2)}_r$, $\mathscr{T}^{(3)}_r$ for $r \geq 2$.*

(4) $\mathscr{D}$ is a critically indecomposable oriented graph iff $\mathscr{D}$ is isomorphic to $\mathscr{P}_r$, $\mathscr{P}'_r$, $\mathscr{T}_r^{(1)}$, $\mathscr{T}_r^{(2)}$, $\mathscr{T}_r^{(3)}$ or $\mathscr{G}_r$ for $r \geq 2$.

Theorem 5.9. *Let $(A, \equiv)$ be indecomposable of order ≥ 7. Then there are distinct $c, d \in A$ such that $A \setminus \{c, d\}$ is indecomposable.*

Proof. If $(A, \equiv)$ is critically indecomposable, then it follows from Corollary 2.3 that there are distinct $c, d \in A$ such that $A \setminus \{c, d\}$ is indecomposable. If $(A, \equiv)$ is indecomposable, then it follows from Corollary 2.3 that $A \setminus \{x\}$ is indecomposable for some $x \in A$. Then, as long as $A \setminus \{x\}$ is not critically indecomposable, there is $y \in A \setminus \{x\}$ such that $A \setminus \{x, y\}$ is indecomposable. Therefore, we can assume that $A = B \cup \{x\}$, where $x \notin B$ and $(B, \equiv)$ is critically indecomposable. It will be proved that there are distinct $c, d \in B$ such that $A \setminus \{c, d\}$ is indecomposable. To derive a contradiction, suppose that this is not the case.

Case 1: *$(B, \equiv)$ is the skeleton of $\mathscr{G}_r$, $\mathscr{P}_r$, $\mathscr{P}'_r$ or $\mathscr{B}_r$, where $r \geq 3$.*

Let I be a nontrivial interval of $A \setminus \{a_1, b_1\}$, and let J be a nontrivial interval of $A \setminus \{a_r, b_r\}$. Since $B \setminus \{a_1, b_1\}$ and $B \setminus \{a_r, b_r\}$ are indecomposable, it follows that $I \setminus \{x\}$ and $J \setminus \{x\}$ are trivial intervals of $B \setminus \{a_1, b_1\}$ and $B \setminus \{a_r, b_r\}$ respectively.

First, suppose $I = B \setminus \{a_1, b_1\}$. If $J = B \setminus \{a_r, b_r\}$, then B is a nontrivial interval of A, so A would not be indecomposable. Thus, we can assume that $J = \{x, y\}$ for some $y \in B \setminus \{a_r, b_r\}$. Then we see that the only possibilities for y are that $y = a_1$ and $r \geq 4$, or $y \in \{a_1, a_2, b_2\}$ and $r = 3$. In any case a contradiction ensures:

If $y = a_1$, then $\{a_1, x\}$ is an interval of A.

If $y = b_2$ and $r = 3$, then $A \setminus \{a_1, b_2\}$ is indecomposable.

If $y = a_2$ and $r = 3$, then $A \setminus \{b_2, a_3\}$ is indecomposable.

Therefore, we can assume $I = \{x, y\}$ and $J = \{x, z\}$, where $y \in B \setminus \{a_1, b_1\}$ and $z \in B \setminus \{a_r, b_r\}$. Then, whenever $w \in B \setminus \{a_1, b_1, a_r, b_r, y, z\}$, then $wy \equiv wx \equiv wz$. Hence, either $y = z$ or else $r = 3$ and $\{y, z\} = \{a_2, b_2\}$. But if $y = z$, then $\{x, y\}$ is an interval of A. So assume $r = 3$ and $\{y, z\} = \{a_2, b_2\}$. Then, if $y = a_2$ and $z = b_2$, then $A \setminus \{a_1, b_3\}$ is indecomposable; and if $y = b_2$ and $z = a_2$; then $A \setminus \{a_2, b_2\}$ is indecomposable.

Case 2: *$(B, \equiv)$ is the skeleton of $\mathscr{T}_r^{(1)}$, where $r \geq 3$.*

Consider some $i \leq 2r$. Then there are nontrivial intervals $I \subseteq A \setminus \{a_i, a_{i+r}\}$ and $J \subseteq A \setminus \{a_{i+1}, a_{i+r+1}\}$. As both $B \setminus \{a_i, a_{i+r}\}$ and $B \setminus \{a_{i+1}, a_{i+r+1}\}$ are isomorphic to skeletons of $\mathscr{T}_{r-1}^{(1)}$, it follows that $I \setminus \{x\}$ and $J \setminus \{x\}$ are trivial intervals of $B \setminus \{a_i, a_{i+r}\}$ and $B \setminus \{a_{i+1}, a_{i+r+1}\}$ respectively.

We will show that $I = \{x, y\}$ for some $y \in B \setminus \{a_i, a_{i+r}\}$. If not, then $I = B \setminus \{a_i, a_{i+r}\}$. Then, if $J = B \setminus \{a_{i+1}, a_{i+r+1}\}$, then B would be a nontrivial interval of A, so that $J = \{x, a\}$ for some $z \in B \setminus \{a_{i+1}, a_{i+r+1}\}$. But then we easily get $p, q \in B \setminus \{a_i, a_{i+r}, a_{i+1}, a_{i+r+1}, z\}$ such that $pz \not\equiv qz$, which contradicts that $pz \equiv px \equiv qx \equiv qz$.

We have shown that for each $i \leq 2r$ there is $y \in B \setminus \{a_i, a_{i+r}\}$ such that $\{x, y\}$ is an interval of $A\setminus\{a_i, a_{i+r}\}$. Thus there is at most one $i \leqslant 2r$ such that $a_i x \not\equiv a_0 a_1 \not\equiv x a_i$. If there is such an i, then clearly $\{x, a_i\}$ is an interval of A. Therefore, we conclude that $(A, \equiv)$ is the skeleton of a tournament, so let us suppose that $(A, \rightarrow)$ is that tournament, and that $(B, \rightarrow)$ is $\mathcal{T}_r^{(1)}$.

Now, since $(A, \rightarrow)$ is indecomposable, there are $i, j \leqslant 2r$ such that $a_i \rightarrow x$, $x \rightarrow a_{i+1}$, $x \rightarrow a_j$ and $a_{j+1} \rightarrow x$, where $i + r \neq j$ and $j + r \neq i$. Without loss of generality, assume $j = i + k$, where $k < r$. Then, if $k > 1$, then we easily see that $A\setminus\{a_{i+1}, a_{i+r+1}\}$ is indecomposable; and if $k = 1$, we easily see that $A\setminus\{a_{j+2}, a_{j+r+2}\}$ is indecomposable.

Case 3: $(B, \equiv)$ *is the skeleton of* $\mathcal{T}_r^{(2)}$, $\mathcal{D}_r$ *or* $\mathcal{D}_r'$, *where* $r \geqslant 3$.

Let I be a nontrivial interval of $A\setminus\{a_0, b_1\}$, and let J be a nontrivial interval of $A\setminus\{a_r, b_r\}$. Then, as in Case 1, we see that there is $y \in B\setminus\{a_0, b_1\}$ such that $I = \{x, y\}$, and there is $z \in B\setminus\{a_r, b_r\}$ such that $J = \{x, z\}$, and $y \neq z$ as otherwise I would be an interval of A. There is $w \in B\setminus\{a_0, b_1, a_r, b_r, y, z\}$ such that $wy \not\equiv wz$. But then $wy \equiv wx \equiv wz$, a contradiction.

Case 4: $(B, \equiv)$ *is the skeleton of* $\mathcal{T}_r^{(3)}$, *where* $r \geqslant 3$.

Let I be a nontrivial interval of $A\setminus\{a_1, a_2\}$, and let J be a nontrivial interval of $A\setminus\{a_{2r-1}, a_{2r}\}$. If $I = B\setminus\{a_1, a_2\}$, then it is easily checked that $J = \{a_1, x\}$, which implies that $\{a_2, x\}$ is an interval of A. Consequently, $I = \{x, y\}$ for some $y \in B\setminus\{a_1, a_2\}$, and $J = \{x, z\}$ for some $z \in B\setminus\{a_{2r-1}, a_{2r}\}$, and $y \neq z$ as otherwise I would be an interval of A. There is $w \in B\setminus\{a_1, a_2, a_{2r-1}, a_{2r}, y, z\}$ such that $wy \not\equiv wz$. But then $wy \equiv wx \equiv wz$, a contradiction. $\square$

The following is an immediate corollary to the previous theorem.

Corollary 5.10. *Suppose $\mathcal{A}$ is an indecomposable structure of order n which is not critically indecomposable, and suppose $5 \leqslant m \leqslant n$. Then $\mathcal{A}$ has an indecomposable substructure of order m.*

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