ELSEVIER

Discrete Applied Mathematics 50 (1994) 111–123

DISCRETE APPLIED MATHEMATICS

# On the poset of all posets on $n$ elements

Richard A. Brualdi[a,*], Hyung Chan Jung[a], William T. Trotter Jr[b]

[a] *Mathematics Department, University of Wisconsin-Madison, Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706, USA*

[b] *Bell Communications Research, 445 South Street, Morristown, NJ 07960, USA*

Received 27 August 1991; revised 18 May 1992

**Abstract**

We consider the poset of all posets on $n$ elements where the partial order is that of inclusion of comparabilities. We discuss some properties of this poset concerning its height, width, jump number and dimension. We also give algorithms to construct some maximal chains in this poset which have special properties for these parameters.

*Key words:* Jump number; Dimension; Width; Height; Power poset

## 1. Introduction

Let $P$ be a finite poset and let $|P|$ be the *number of vertices* in $P$. A *subposet* is a subset of $P$ with the induced order. A *chain* $C$ in $P$ is a subposet of $P$ which is a linear order. The *length* of the chain $C$ is $|C| - 1$. If $a$ and $b$ are in $P$, then $b$ covers $a$, written $a \prec b$, provided that $a < c \leqslant b$ implies that $c = b$. A *saturated* chain is a chain of the form $x_0 \prec x_1 \prec \cdots \prec x_k$. If $x, y \in P$ satisfy $x < y$ then the pair $(x, y)$ is called a *comparability* of $P$. A poset is *ranked* if every maximal chain has the same length. A *linear extension* of a poset $P$ is a linear order $L = x_1, x_2, \ldots, x_n$ of the elements of $P$ such that $x_i < x_j$ in $P$ implies $i < j$. Szpilrajn [17] showed that the set $\mathscr{L}(P)$ of all linear extensions of $P$ is not empty.

Let $P, Q$ be two disjoint posets. The *disjoint sum* $P + Q$ of $P$ and $Q$ is the poset on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$ and $x < y$ in $P$ or $x, y \in Q$ and $x < y$ in $Q$. The *linear sum* $P \oplus Q$ of $P$ and $Q$ is obtained from $P + Q$ by adding the relation $x < y$ for all $x \in P$ and $y \in Q$.

Throughout this section, $L$ denotes an arbitrary linear extension of $P$. A $(P, L)$-*chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \cdots \prec z_k$ in both $L$ and $P$. Let $c(L)$ be the number of $(P, L)$-chains in $L$.

*Corresponding author.


*SSDI* 0166-218X(92)00169-Q

A consecutive pair $(x_i, x_{i+1})$ of elements in $L$ is a *jump* of $P$ in $L$ if $x_i$ is not comparable to $x_{i+1}$ in $P$. The jumps induce a decomposition $L = C_1 \oplus \cdots \oplus C_m$ of $L$ into $(P, L)$-chains $C_1, \ldots, C_m$, where $m = c(L)$ and $(\max C_i, \min C_{i+1})$ is a jump of $P$ in $L$ for $i = 1, \ldots, m-1$. Let $s(L, P)$ be the number of jumps of $P$ in $L$. The *jump number* $s(P)$ of $P$ is the minimum of $s(L, P)$ over all linear extensions of $L$ of $P$. If $s(L, P) = s(P)$ then $L$ is called an *optimal linear extension* of $P$. We denote the set of al optimal linear extensions of $P$ by $\mathcal{O}(P)$.

The *width* $\omega(P)$ of $P$ is the maximal number of elements of an *antichain* (mutually incomparable elements) of $P$. Dilworth [4] showed that $\omega(P)$ equals the minimum number of chains in a partition of $P$ into chains. Since for any linear extension $L$ of $P$ the number of $(P, L)$-chains is at least as large as the minimum number of chains in a chain partition of $P$, it follows from Dilworth's theorem that

$$s(P) \geqslant \omega(P) - 1. \tag{1}$$

If equality holds in (1), then $P$ is called a *Dilworth poset* or simply a *D-poset*. More discussion about D-posets is given in [5, 16].

A *cycle* is a partially ordered set with diagram in Fig. 1(a) where $n \geqslant 2$. In 1982, Duffus et al. [5] proved the following lemma.

**Lemma 1.1.** *A poset which does not have a subposet isomorphic to a cycle is a D-poset.*

It follows that if $P$ is a D-poset, then for every optimal linear extension $L$ the $(P, L)$-chains in $L$ form a minimum chain partition of $P$.

A linear extension $L = x_1, x_2, \ldots, x_n$ of $P$ is *greedy* if $L$ can be obtained by applying the following algorithm:

*Step* 1. Choose a minimal element $x_1$ of $P$.

*Step* 2. Suppose $x_1, \ldots, x_i$ have been chosen. If there is at least one minimal element of $P \setminus \{x_1, \ldots, x_i\}$ which is greater than $x_i$ then choose $x_{i+1}$ to be any such minimal element; otherwise, choose $x_{i+1}$ to be any minimal element of $P \setminus \{x_1, \ldots, x_i\}$.

Let $\mathcal{G}(P)$ be the set of all greedy linear extensions of $P$. In Fig. 2, $L_1, L_2, L_3$ are greedy linear extensions of the poset $N$, but $L_4$ is not greedy. Only $L_3$ is optimal. So $\mathcal{O}(N) \subseteq \mathcal{G}(N)$. In fact, $L_3$ is a greedy optimal linear extension of $N$. Since the greedy algorithm above is a particular way of carrying out the algorithm for a linear extension, every poset $P$ has a greedy linear extension. As remarked in [3, 15] every poset has an optimal greedy linear extension.

A poset $P$ is *greedy* if $\mathcal{G}(P) \subseteq \mathcal{O}(P)$, that is, every greedy extension is optimal.

A poset $P$ is *N-free* if $P$ contains no cover-preserving subposet isomorphic to the poset $N$ in Fig. 2. The following lemma in [14] partially characterizes greedy posets.

**Lemma 1.2.** *Every N-free poset is greedy.*

The next lemma is from [6].

**Lemma 1.3.** *A poset which does not contain a subposet isomorphic to $C$ in Fig.* 1(b) *satisfies $\mathcal{O}(P) \subseteq \mathcal{G}(P)$.*

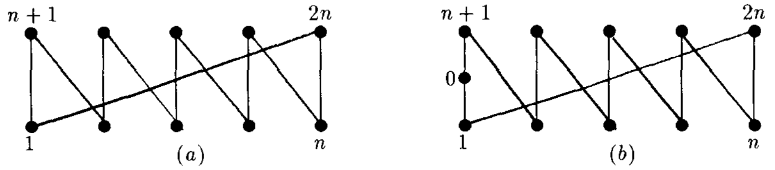


Fig. 1. [$n \geqslant 2$]: (a) Crown, (b) C.

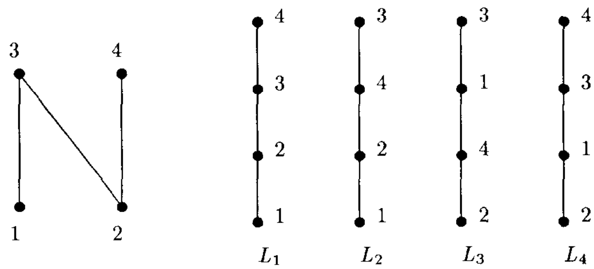


Fig. 2. The poset $N$ and its linear extensions.

The *dual* of the poset $P$ is the poset $P^{\mathrm{d}}$ obtained from $P$ by reversing the order. A poset $P$ is said to be *reversible* if $L^{\mathrm{d}} \in \mathscr{G}(P^{\mathrm{d}})$ for every $L \in \mathscr{G}(P)$. Rival and Zaguia [15] showed the following lemma.

**Lemma 1.4.** *A poset $P$ is reversible if and only if $\mathcal{O}(P) = \mathscr{G}(P)$.*

Szpilrajn [17] also proved that any order relation is the intersection of its linear extensions. A set of linear extensions of $P$ whose intersection is $P$ is called a *realizer* of $P$. The *dimension* of an ordered set $P$ is the minimum cardinality of a realizer of $P$. A *minimum realizer* of $P$ is a realizer which achieves the dimension of $P$.

Let $P$ be a poset and let $x$ be a point of $P$. It follows easily that the width function satisfies

$$\omega(P) \geqslant \omega(P \backslash \{x\}) \geqslant \omega(P) - 1.$$

Call a poset $P$ *width-critical* if $\omega(P\backslash\{x\}) < \omega(P)$ for all $x \in P$. It is clear that $P$ is width-critical if and only if $P$ is an antichain.

The *height* $h(P)$ of $P$ is the length of the longest chain in $P$. It also follows easily that

$$h(P) \geqslant h(P\backslash\{x\}) \geqslant h(P) - 1.$$

Call a poset $P$ *height-critical* if $h(P\backslash\{x\}) < h(P)$ for each $x \in P$. Clearly, $P$ is height-critical if and only if $P$ is a linear order.

Another simple fact is that

$$s(P) \geqslant s(P\backslash\{x\}) \geqslant s(P) - 1.$$

Call a poset $P$ *jump-critical* if $s(P\backslash\{x\}) < s(P)$ for each $x \in P$. Jump-critical posets, however, are much more complicated and not well understood. El-Zahar and Schmer

[8] showed that a jump-critical poset $P$ with jump number $m$ has at most $(m + 1)!$ elements. El-Zahar and Rival [7] showed that there are precisely jump-critical posets with jump number at most three.

The following theorem is due to Híraguchi [9].

**Theorem 1.5.** *Removing one point from a poset decreases its dimension by at most one.*

Clearly, the removal of a point $x$ from a poset $P$ cannot increase its dimension. Hence, we have

$$\dim P \geqslant \dim(P \setminus \{x\}) \geqslant \dim P - 1.$$

Call a poset $P$ (of dimension $d$) *dimension-critical* ($d$-irreducible) provided $\dim(P \setminus \{x\}) < \dim P$ for all points $x$ of $P$. The 3-irreducible posets have been characterized by Kelley [10] and Trotter and Moore [19], but no characterization is known for the $d$-irreducible posets for $d \geqslant 4$. It is known that for each $d \geqslant 3$ there exist infinitely many dimension-critical posets of dimension $d$.

In summary, removing one point from a poset does not increase any of the parameters width, height, jump number and dimension and can decrease each of them by at most one. It is natural to consider the effect of removing one comparability on these parameters.

The removal of one comparability does not in general result in a poset. Only a comparability which cannot be recovered by transitivity can be removed. Thus, removing the comparability $x < y$ results in a poset if and only if $x \prec y$, that is, $y$ covers $x$. Similarly, the addition of one comparability does not in general result in a poset. Only a comparability which does not 'force' other comparabilities can be added. Thus, the comparability $x < y$ can be added to $P$ with the result being a poset (with exactly one more comparability) if and only if $u < x$ in $P$ implies $u < y$, and $y < v$ in $P$ implies $x < v$. Such a pair $(x, y)$ is usually called a *nonforcing ordered pair* of $P$ [13].

Let $[n]$ be $\{1, 2, \ldots, n\}$. Let $P$ and $Q$ be posets on $[n]$. We define $P \subset Q$ provided that $P \neq Q$ and $Q$ has all the comparabilities that $P$ has. This defines a partial order on the set of all posets on $[n]$. The resulting poset is denoted by $\mathscr{P}_n$ and is called a *power poset*. The minimum element of $\mathscr{P}_n$ is $\emptyset$, i.e, an antichain on $[n]$. The maximal elements of $\mathscr{P}_n$ are the linear orders on $[n]$. Throughout this paper, we let $n^* = \binom{n}{2}$.

## 2. Power posets

Let $P_1$ and $P_2$ be posets in $\mathscr{P}_n$ with $P_1 \subset P_2$. Then there exists a pair of elements $a$ and $b$ such that $a \prec b$ in $P_2$ but $a$ and $b$ are incomparable in $P_1$. Removing the comparability $a \prec b$ (and only this comparability) from $P_2$ results in a poset $P_3$ in $\mathscr{P}_n$ such that $P_1 \subseteq P_3 \subset P_2$. The number of comparable pairs in $P_3$ is one less than the number of comparable pairs of $P_2$. Hence, $\mathscr{P}_n$ is a ranked poset (the rank function is the number of comparable pairs) and thus satisfies the Jordan–Dedekind chain condition. In particular, for any poset $P$ with $|P| = n$, there exist a maximal chain $\mathscr{C}$ of length $n^*$ which contains $P$ in $\mathscr{P}_n$.

Let $\mathscr{C}$ be a maximal chain in $\mathscr{P}_n$. The *height sequence* $H(\mathscr{C})$ of $\mathscr{C}$ is $(h_0(\mathscr{C}), \ldots, h_{n^*}(\mathscr{C}))$ where $h_i(\mathscr{C})$ is the height of the poset in $\mathscr{C}$ with $i$ comparabilities. Similarly, the *width sequence* of $\mathscr{C}$ is $W(\mathscr{C}) = (\omega_0(\mathscr{C}), \ldots, \omega_{n^*}(\mathscr{C}))$, the *jump number sequence* is $J(\mathscr{C}) = (s_0(\mathscr{C}), \ldots, s_{n^*}(\mathscr{C}))$, and the *dimension sequence* is $D(\mathscr{C}) = (d_0(\mathscr{C}), \ldots, d_{n^*}(\mathscr{C}))$. For instance, for the maximal chain $\mathscr{H}_6^{\ell}$ in $\mathscr{P}_6$ given in Fig. 3 we have

$$H(\mathscr{H}_6^{\ell}) = (0,1,1,1,1,1,1,1,1,1,2,2,3,3,4,5),$$

$$W(\mathscr{H}_6^{\ell}) = (6,5,5,4,4,3,3,3,3,3,3,3,2,2,2,1),$$

$$J(\mathscr{H}_6^{\ell}) = (5,4,4,3,3,2,3,3,3,4,3,3,2,2,1,0),$$

$$D(\mathscr{H}_6^{\ell}) = (2,2,2,2,2,2,3,2,2,2,2,2,2,2,2,1).$$

In this section, we consider the change in $\mathscr{P}_n$ of the four basic parameters as one comparability is added or deleted. Since adding one comparability is the reverse of removing one comparability, it suffices to consider only the removal of one comparability.

By direct observation, we get the change of height and width in $\mathscr{P}_n$.

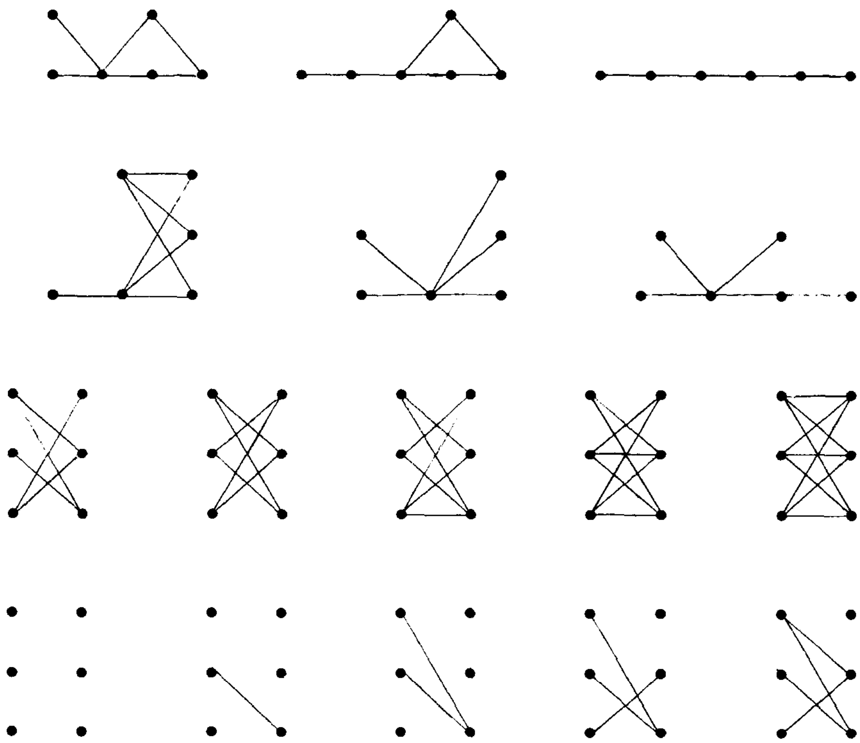

Fig. 3. Maximal chain $\mathscr{H}_6^{\ell}$ listed from top to bottom and left to right.

**Proposition 2.1.** *If $Q \prec P$ in $\mathscr{P}_n$, then*

$$h(P) - 1 \leqslant h(Q) \leqslant h(P)$$

*and*

$$\omega(P) \leqslant \omega(Q) \leqslant \omega(P) + 1.$$

**Proof.** For height, the result is clear. Let $Q$ be a poset obtained from $P$ by deleting one comparability $(a, b)$. Clearly, $\omega(Q) \geqslant \omega(P)$. Since for any antichain $A$ of $Q$, either $A$ or $A \setminus \{a\}$ is an antichain of $P$, we also have $\omega(Q) \leqslant \omega(P) + 1$. □

**Corollary 2.2.** *For any finite poset $P$ with $n$ elements, choose any saturated chain $P = P_0 \prec P_1 \prec \cdots \prec P_m$ in $\mathscr{P}_n$ such that $P_m$ is a linear extension of $P$. Then*

$$h(P_0) \leqslant h(P_1) \leqslant \cdots \leqslant h(P_m)$$

*and*

$$\omega(P_0) \geqslant \omega(P_1) \geqslant \cdots \geqslant \omega(P_m).$$

A *URT-poset* is a poset whose Hasse diagram is an upward rooted tree possibly with some isolated points. Both linear orders and antichains are URT-posets. The following result is due to Wolk [20]; we outline a simple inductive proof.

**Lemma 2.3.** *Let $P$ be a URT-poset. Then* $\dim P = 2$ *if and only if $P$ is not a linear order.*

**Proof.** A linear order has dimension 1. Adding isolated points to a poset of dimension at least 2 does not change the dimension. Hence, to complete the proof it suffices to show that the dimension of a poset $T$ which is not a linear order but whose Hasse diagram is an upward rooted tree equals 2. Let $x$ be the last point at which $T$ branches into (disjoint) rooted trees $T_1, \ldots, T_k$ ($k \geqslant 2$). By induction, $\dim T_i \leqslant 2$ for each $i$. Moreover, by induction the dimension of the rooted subtree $T'$ obtained from $T$ by deleting all but the root of each of $T_1, \ldots, T_k$ equals 2. It is now easy to use minimum realizers of $T'$, $T_1, \ldots, T_k$ to obtain two linear orders whose intersections is $T$. □

We now give algorithms for constructing some special maximal chains of $\mathscr{P}_n$.

**Algorithm 2.4.** *Construction of $\mathscr{H}_n^{\mathrm{e}}$.*

Algorithm E-Height($n$):
$l \leftarrow 0$
$R(l) \leftarrow n$ vertices with no comparability
for $j = 2$ to $n$ do
  for $i = 1$ to $j - 1$ do
    $l \leftarrow l + 1$
    $R(l) \leftarrow$ add $(i, j)$ to $R(l - 1)$
$\mathscr{H}_n^{\mathrm{e}} \leftarrow \{R(l)\colon l = 0, 1, \ldots, n^*\}$.

A *standard poset* $S_n$ is defined on $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ as follows: all $a_i$'s are minimal, all $b_j$'s are maximal, and $a_i < b_j$ for all $i \neq j$. It is well known [18] that $\dim S_n = n$. A *pseudo-standard poset* $S_n^*$ is defined to be $S_n \oplus \{x\}$ where $x \notin S_n$. We also have $\dim S_n^* = n$.

A *complete bipartite poset* $K_{p,q}$ is defined on $\{a_1, \ldots, a_p, b_1, \ldots, b_q\}$ as follows: all $a_i$'s are minimal, all $b_j$'s are maximal, and $a_i < b_j$ for all $i, j$.

**Algorithm 2.5.** *Construction of* $\mathscr{H}_n^{\ell}$.

Algorithm L-Height($n$):

Let $I_0 = \{(1, n)\}$. Proceeding recursively, for $i = 1, \ldots, \lceil \log_2 n \rceil$, let $I_i$ be the set of integer intervals consisting of $(p, \lceil (p+q)/2 \rceil - 1)$, $(\lceil (p+q)/2 \rceil, q)$ for each $(p, q) \in I_{i-1}$. If $a > b$ then $(a, b) = \emptyset$. For $(a, b)$ and $(c, d)$ in $I_i$, we define $(a, b) \leqslant (c, d)$ if and only if either (i) $(b - a) > (d - c)$ or (ii) $(b - a) = (d - c)$ and $a > c$ holds.

```
Procedure Complete (p, q, t);
If q = p then {do nothing}
else
begin {else}
  k ← ⌈(p + q)/2⌉ − 1
  for i = p to k do {Construct standard poset}
    for j = k + 1 to 2k − p + 1 do
      if j − i = k − p + 1 then {do nothing}
      else
      begin {else}
        t ← t + 1
        R(t) ← add (i, j) to R(t − 1)
      end {else}
  for i = p to k do {Construct complete bipartite poset}
    t ← t + 1
    R(t) ← add (i, i + k − p + 1) to R(t − 1)
  if q − p + 1 is odd then
    for i = p to k do
      t ← t + 1
      R(t) ← add (i, q) to R(t − 1)
  else {do nothing}
end {else}
begin {Main}
  t ← 0
  R(t) ← n vertices with no comparability
  for i = 0 to ⌈log₂ n⌉ do
    J ← I_i
    repeat
      choose the smallest element (a, b) ∈ J satisfying a < b
      Complete(a, b, t)
      J ← J \ {(a, b)}
```

*until* ( $J = \emptyset$) or (every element $(c,d) \in J$ satisfies $c = d$)
$\mathscr{H}_n^{\ell} \leftarrow \{R(l)\colon l = 0, 1, \ldots, n^*\}$
end {Main}

We have the following:

(1) $\mathscr{H}_n^{e}$ and $\mathscr{H}_n^{\ell}$ are maximal chains of length $n^*$.

(2) Each of the posets in $\mathscr{H}_n^{e}$ are UTR-posets and hence by Lemma 2.3, we have $D(\mathscr{H}_n^{e}) = (2, \ldots, 2, 1)$. Moreover, in $\mathscr{H}_n^{e}$ height increases as early as possible.

(3) In $\mathscr{H}_n^{\ell}$ height increases as late as possible as one comparability is added.

(4) $\mathscr{H}_n^{\ell}$ contains a poset of maximum possible dimension $\lfloor n/2 \rfloor$.

A *subdivision* of a poset $P$ is a poset whose Hasse diagram is obtained from that of $P$ by subdividing its edges by the insertion of new points. We define the four point poset $\diamond$ to be the poset $1 \oplus (1 + 1) \oplus 1$. We denote by $P^*$ any poset which is obtained from $P$ by putting a point above one of its maximal elements. The following lemma follows as a special case of a theorem of Hiraguchi [9] but we include its simple proof.

**Lemma 2.6.** *The dimension of a poset which is a disjoint sum of chains, URT-posets, subdivisions of $N$'s, $N^*$'s, $\diamond$'s and $\diamond^*$'s is at most* 2.

**Proof.** Each of $N, N^*$, $\diamond$ and $\diamond^*$ has dimension 2. By Lemma 2.3, a URT-poset has dimension 2. Since subdividing these particular posets does not change their dimensions, the lemma follows. □

**Algorithm 2.7.** *Construction of $\mathscr{W}_n^{e}$.*

Algorithm E-Width($n$):
  Step 0 {Initialize}
$l \leftarrow 0$
$R(l) \leftarrow n$ vertices with no comparability
Step 1. {Construct two-element chains}
*repeat*
  $l \leftarrow l + 1$
  construct a chain by adding one comparability
  $R(l) \leftarrow$ add above comparability to $R(l-1)$
*until* at most one vertex remains
  Step 2. {Merge two chains}
Choose two chains $\mathscr{C}_1$, $\mathscr{C}_2$ of the smallest size
*repeat*
  $l \leftarrow l + 1$
  add one comparability between $\mathscr{C}_1$ and $\mathscr{C}_2$ without destroying tree diagram
  $R(l) \leftarrow$ add above comparability to $R(l-1)$
*until* $\mathscr{C}_1$ and $\mathscr{C}_2$ merge into one chain
  Step 3. {Terminal condition}
If $R(l)$ is a linear extension then go to step 4; otherwise, go to step 2.
  Step 4 {Define $\mathscr{W}_n^{e}$}
$\mathscr{W}_n^{e} \leftarrow \{R(l)\colon l = 0, 1, \ldots, n^*\}$.

**Algorithm 2.8.** *Construction of* $\mathscr{W}_n^\ell$.

Algorithm L-Width($n$):
$l \leftarrow 0$
$R(l) \leftarrow n$ vertices with no comparability
for $i = 1$ to $n - 1$ do
  for $j = i + 1$ to $n$ do
    $l \leftarrow l + 1$
    $R(l) \leftarrow$ add $(i,j)$ to $R(l-1)$
$\mathscr{W}_n^\ell \leftarrow \{R(l): l = 0, 1, \ldots, n^*\}$

We have the following:

(1) $\mathscr{W}_n^e$ and $\mathscr{W}_n^\ell$ are maximal chains of length $n^*$.

(2) As one comparability is added, width decreases as early as possible in $\mathscr{W}_n^e$, and width decreases as late as possible in $\mathscr{W}_n^\ell$.

(3) Each of the posets in $\mathscr{W}_n^\ell$ is a URT-poset and so $D(\mathscr{W}_n^\ell) = (2, \ldots, 2, 1)$. Each of the posets $\mathscr{W}_n^e$ satisfies the hypothesis of Lemma 2.6 and hence $D(\mathscr{W}_n^e) = (2, \ldots, 2, 1)$.

**Theorem 2.9.** *If* $Q \prec P$ *in* $\mathscr{P}_n$, *then*

$$s(P) - 1 \leqslant s(Q) \leqslant s(P) + 1.$$

**Proof.** Let $Q$ be a poset obtained from $P$ by deleting one comparability $(x, y)$.

Choose any $L_P$ in $\mathcal{O}(\mathscr{P})$. Then $L_P \in \mathscr{L}(Q)$. Now $s(Q) \leqslant s(L_P, Q) \leqslant s(L_P, P) + 1 = s(P) + 1$. Thus, $s(Q) \leqslant s(P) + 1$.

Suppose that $s(Q) < s(P) - 1$. Choose any $L_Q \in \mathcal{O}(Q)$. If $x < y$ in $L_Q$, then $L_Q \in \mathscr{L}(P)$ since $y$ covers $x$ in $P$. Thus, $s(Q) = s(L_Q, Q) \geqslant s(L_Q, P) \geqslant s(P)$, contradicting $s(Q) < s(P) - 1$. Hence, $y < x$ in $L_Q$, and so $L_Q \notin \mathscr{L}(P)$. Now we construct $L$ as follows: *delete* $x$ *from* $L_Q$ *and put* $x$ *just below* $y$ *in* $L_Q \setminus \{x\}$. If $a < x$ in $Q$, then $a < x$ in $P$; thus, by transitivity $a < y$ in $P$ and so in $Q$. Then $L \in \mathscr{L}(Q)$. Since $y > x$ in $L$ and all other comparabilities of $Q$ are preserved in $L$, we have $L \in \mathscr{L}(P)$. If $s(L, P) \leqslant s(L_Q, Q) = s(Q)$, then $s(P) \leqslant s(Q)$, contradicting $s(Q) < s(P) - 1$. Thus,

$$s(L, P) \geqslant s(L_Q, Q) + 1. \tag{2}$$

Let $z^* = \sup\{z \in L_Q: z < y \text{ in } L_Q\}$, $w_* = \inf\{w \in L_Q: w > x \text{ in } L_Q\}$, and $t^* = \sup\{t \in L_Q: t < x \text{ in } L_Q\}$. Let $C_x$ be a $(Q, L_Q)$-chain which contains $x$.

*Case* 1: $C_x = \{x\}$. If $w_* > t^*$ in $Q$, then choose a $(Q, L_Q)$-chain $C$ which contains $t^*$. Put $x$ just below $\inf C$ and get a new linear extension $L'_Q$ of $Q$. Since $s(L'_Q, Q) = s(L_Q, Q) - 1 = s(Q) - 1$, we have $s(L'_Q, Q) < s(Q)$, contradicting $L_Q$ is an optimal linear extension of $Q$. Thus, $w_*$, $t^*$ are incomparable in $Q$. Now we get $s(L_Q, Q) \geqslant s(L, P)$, and thus $s(P) \leqslant s(Q)$ contradicting $s(Q) < s(P) - 1$.

*Case* 2: $C_x$ has at least two elements. Choose any $u \in C_x \setminus \{x\}$. If $u < x$ in $L_Q$, then $u < x$ in $C_x$. So $u < x$ in $Q$, and $u < y$ in $L_Q$. However, $\inf C_x > y$ in $L_Q$, and by transitivity $\inf C_x > u$ in $L_Q$, which contradicts $u \in C_x$. Thus, $x = \inf C_x$. If neither $(z^*, x)$ nor $(t^*, w_*)$ is a jump of $P$ in $L$, then $s(L, P) \leqslant s(L_Q, Q)$, contradicting (2).

Hence, either $(z^*, x)$ or $(t^*, w_*)$ is a jump of $P$ in $L$. Thus, $s(L, P) \leqslant s(L_Q, Q) + 1 = s(Q) + 1$, and so $s(P) \leqslant s(Q) + 1$ which contradicts $s(Q) < s(P) - 1$.
Therefore, we get $s(P) - 1 \leqslant s(Q)$. □

Pouzet and Rival [12] showed that the dimension behaves in a similar way, that is, if $P$ and $Q$ are in $\mathscr{P}_n$ with $Q \prec P$, then

$$\dim P - 1 \leqslant \dim Q \leqslant \dim P + 1.$$

## 3. Concluding remarks

A sequence $(a_0, \dots, a_m)$ has the *monotone decreasing property* if $a_0 \geqslant \cdots \geqslant a_m$.

**Lemma 3.1.** *$J(\mathscr{W}_n^e)$ and $J(\mathscr{W}_n^\ell)$ have the monotone decreasing property.*

**Proof.** By construction no poset in $\mathscr{W}_n^e$ contains a cycle as a subposet. So by Lemma 1.1, $s(P) = \omega(P) - 1$ for every $P$ in $\mathscr{W}_n^e$. By Proposition 2.1, $\omega_i(\mathscr{W}_n^e) \geqslant \omega_{i+1}(\mathscr{W}_n^e)$ for all $i$. Hence $s_i(\mathscr{W}_n^e) \geqslant s_{i+1}(\mathscr{W}_n^e)$ for all $i$. Thus, $J(\mathscr{W}_n^e)$ has the desired property. In a similar way, we see that $J(\mathscr{W}_n^l)$ has the desired property. □

Lemma 3.1 suggests the following result.

**Theorem 3.2.** *For any finite poset $P$ with $n$ elements there is a saturated chain $P = P_0 \prec P_1 \prec P_2 \prec \cdots \prec P_m$ in $\mathscr{P}_n$ such that $P_m$ is a linear extension of $P$ and $s(P_0) \geqslant s(P_1) \geqslant s(P_2) \geqslant \cdots \geqslant s(P_m)$.*

**Proof.** Let $s(P) = t$. If $t = 0$ then $P$ is a linear order and the theorem holds (with $m = 0$). Thus, we suppose that $t \geqslant 1$. It suffices to show that there is a poset $Q$ such that $P \prec Q$ and $s(P) \geqslant s(Q)$. Let $L$ be a linear extension of $P$ with $s(L, P) = t$. There exists a nonforcing pair $(a, b)$ of $P$ such that $a$ comes before $b$ in $L$. Let $Q$ be the poset obtained from $P$ by adding the comparability $a < b$. Then $P \prec Q$ and $L$ is also a linear extension of $Q$. We have

$$s(P) = s(L, P) \geqslant s(L, Q) \geqslant s(Q). \qquad \square$$

Pouzet and Rival [12] conjectured that a result similar to Theorem 3.2 holds for the dimension. We now verify this conjecture. (Algorithms 2.4, 2.7 and 2.8 show that this conjecture holds for $P = \emptyset$.)

**Theorem 3.3.** *For any finite poset $P$ with $n$ elements there is a saturated chain $P = P_0 \prec P_1 \prec \cdots \prec P_m$ in $\mathscr{P}_n$ such that $P_m$ is a linear extension of $P$ and $\dim P_0 \geqslant \dim P_1 \geqslant \cdots \geqslant \dim P_m$.*

**Proof.** We may assume that $P$ is not a linear order. It suffices to show that there is a poset $Q$ with $P \prec Q$ such that $\dim P \leqslant \dim Q$. Let $\{L_1, \dots, L_t\}$ be a minimum realizer of $P$. Then there exists an nonforcing pair $(a, b)$ in $P$ such that $a$ precedes $b$ in

$L_i$ ($i = 1, \ldots, t-1$) and $b$ precedes $a$ in $L_t$. We may choose such a pair $(a, b)$ so that $a$ and $b$ are as close as possible in $L_t$. Let $Q$ be the poset obtained from $P$ by adding the comparability $a < b$, and let $L_t'$ be the linear order obtained from $L_t$ by moving $a$ so that $a$ is immediately below $b$. Then $P \prec Q$ and $\{L_1, \ldots, L_{t-1}, L_t'\}$ is a realizer for $Q$. Hence, $\dim P = t \geqslant \dim Q$. $\square$

One may ask whether or not for any maximal chain $\mathscr{C}$ in $\mathscr{P}_n$, $\omega_i(\mathscr{W}_n^{\mathrm{e}}) \leqslant \omega_i(\mathscr{C})$ is true for all $i$. This is false. For example, let $\mathscr{C}'$ be any maximal chain in $\mathscr{P}_6$ which contains $\underline{3} + \underline{3}$; then $\omega_6(\mathscr{W}_6^{\mathrm{e}}) > \omega_6(\mathscr{C}')$. The corresponding question about $\mathscr{W}_n^{\ell}$ is false. Let $\mathscr{D}$ be a maximal chain which contains $K_{4,4}$. Then $s_{16}(\mathscr{D}) = 6$, $s_{16}(\mathscr{W}_4^{\ell}) = 4$. This shows that there exists a maximal chain $\mathscr{C}$ such that $s_i(\mathscr{C}) \not\leqslant s_i(\mathscr{W}_4^{\ell})$.

**Theorem 3.4.** *Let $\mathscr{C}$ be a maximal chain in $\mathscr{P}_n$. If every poset in $\mathscr{C}$ is $N$-free, then*

(a) *no poset in $\mathscr{C}$ has a subposet isomorphic to a cycle or the poset $C$ (in Fig.* 1(b)).

(b) *$J(\mathscr{C})$ has the monotone decreasing property.*

(c) *for each $P$ in $\mathscr{C}$, there is an $L \in \mathcal{O}(P)$ in which the $(P, L)$-chains in $L$ form a minimum chain partition of $P$.*

(d) *every poset $P$ in $\mathscr{C}$ is reversible.*

**Proof.** (a) Suppose some poset $P$ in $\mathscr{C}$ has a subposet $P'$ isomorphic to a cycle or to $C$. If $P'$ is isomorphic to a cycle with at least 6 points or a $C$, then $P$ has a subposet isomorphic to $N$. If the cycle has 4 points, then since removing any comparability of such a cycle leaves an $N$, some poset $Q \subset P$ in $\mathscr{C}$ contains a subposet isomorphic to $N$. In either case we obtain a contradiction.

(b) Since no poset in $\mathscr{C}$ contains a subposet isomorphic to a cycle, then by Lemma 1.1, $s_i(\mathscr{C}) = \omega_i(\mathscr{C}) - 1$. Applying Proposition 2.1, we get $s_{i+1}(\mathscr{C}) \leqslant s_i(\mathscr{C})$ for all $i$.

(c) It follows from (a) and Lemma 1.1.

(d) Since $P$ is $N$-free, $\mathscr{G}(P) \subseteq \mathcal{O}(P)$ by Lemma 1.2. By (a), $P$ does not contain a subposet isomorphic to $C$; by Lemma 1.3 $\mathcal{O}(P) \subseteq \mathscr{G}(P)$. Applying Lemma 1.4 we obtain (d). $\square$

Let $I$ denote a chain of length 1. Let $N_0$ be a poset obtained from the union of $N$ and a vertex with no other comparability. Let $V$ be a poset with three elements $\{x, y, z\}$ such that $x > y$ and $z > y$ holds, and $x$ and $z$ are incomparable. Let $V^{\#}$ be a poset obtained from the union of $V$ and two isolated vertices with no other comparabilities.

We have examined four parameters—height, width, jump number, and dimension. However, these four parameters do not determine either a poset or a maximal chain uniquely. For example, $h(N_0) = h(V + I)$, $\omega(N_0) = \omega(V + I)$, $s(N_0) = s(V + I)$, $\dim N_0 = \dim(V + I)$. Let $\mathscr{C}_1$ be a maximal chain in $\mathscr{P}_5$ which contains $W$ (a poset obtained from $V + I$ by letting one of the maximal points of $V$ to be greater than the minimal point of $I$), $N_0$, $V^{\#}$. Let $\mathscr{C}_2$ be a maximal chain obtained from $\mathscr{C}_1$ by replacing $N_0$ by $V + I$. Then $H(\mathscr{C}_1) = H(\mathscr{C}_2)$, $W(\mathscr{C}_1) = W(\mathscr{C}_2)$, $S(\mathscr{C}_1) = S(\mathscr{C}_2)$, $D(\mathscr{C}_1) = D(\mathscr{C}_2)$.

We close with some problems concerning the determination of global properties of $\mathscr{P}_n$. First we note that $h(\mathscr{P}_n) = n^*$.

**Problem 3.5.** *Determine* $\omega(\mathscr{P}_n)$, $\dim \mathscr{P}_n$ *and* $s(\mathscr{P}_n)$.

It is not difficult to obtain an upper and an lower bound for $\dim \mathscr{P}_n$. First partition $[n]$ into two sets $X$ and $Y$ of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, respectively, and consider the *bipartite poset* $P$ with $x < y$ for all $x$ in $X$ and $y$ in $Y$ (and no other comparabilities). Let $\mathscr{P}_n^1$ consist of the $\lfloor n/2 \rfloor \lceil n/2 \rceil$ posets in $\mathscr{P}_n$ each with exactly one of the comparabilities in $P$ and let $\mathscr{P}_n^2$ consist of the $\lfloor n/2 \rfloor \lceil n/2 \rceil$ posets in $\mathscr{P}_n$ each without exactly one of the comparabilities in $P$. Then the subposet of $\mathscr{P}_n$ determined by $\mathscr{P}_n^1 \cup \mathscr{P}_n^2$ is a standard poset and hence we have $\dim \mathscr{P}_n \geqslant \lfloor n/2 \rfloor \lceil n/2 \rceil$. On the other hand, the nonforcing pairs of $\mathscr{P}_n$ are the pairs $(P', P'')$ where $P'$ is a poset with exactly one comparability and $P''$ is a linear order. (This latter fact follows easily from the observation that if $(P', P'')$ is a nonforcing pair then in $\mathscr{P}_n$, $P'$ cannot cover a poset with a nonempty set of comparabilities and dually $P''$ cannot be covered by a poset.) Since $n(n-1)$ partial linear extensions of $\mathscr{P}_n$ suffice to contain all the nonforcing pairs of $\mathscr{P}_n$ we have (see [11]) that $\dim \mathscr{P}_n \leqslant n(n-1)$. Thus, we have

$$\lfloor n/2 \rfloor \lceil n/2 \rceil \leqslant \dim \mathscr{P}_n \leqslant n(n-1).$$

The asymptotic lower bound $\log_2 \omega(\mathscr{P}_n) \geqslant n^2/4$ follows by considering antichains of bipartite posets in $\mathscr{P}_n$.

**Problem 3.6.** *Find properties of* $\mathscr{P}_n$ *that are invariant under the action of the symmetric group on* $[n]$.

**Note added in proof.** The preprint version of this paper contained the assertion in Theorem 3.2 as a conjecture. Theorem 3.2 and Theorem 3.3 appear as Proposition 6 and Corollary 5 in "A generalized permutohedron" by M. Pouzet, K. Reuter, I. Rival, and N. Zaguia, to appear in *Algebra Universalis.*

## References


[1] M.D. Atkinson, The complexity of orders, in: I. Rival, ed., Algorithms and Order (1989) 195–230.

[2] M. Chein and M. Habib, The jump number of dags and posets: an introduction, Ann. Discrete Math. 9 (1980) 189–194.

[3] O. Cogis, Problem 4.6, in: I. Rival, ed., Ordered Sets (Reidel, Dordrecht, 1982) 814.

[4] R.P. Dilworth, A decomposition theorem for partially ordered sets, Ann. Math. 51 (1950) 161–166.

[5] D. Duffus, I. Rival and P. Winkler, Minimizing setups for cycle-free ordered sets, Proc. Amer. Math. Soc. 85 (1982) 509–513.

[6] M.H. ElZahar and I. Rival, Greedy linear-extensions to minimize jumps, Discrete Appl. Math. 11 (1985) 143–156.

[7] M.H. ElZahar and I. Rival, Examples of jump-critical ordered sets, SIAM J. Algebra Discrete Mech. 6 (1985) 713–720.

[8] M.H. ElZahar and J.H. Schmerl, On the size of jump-critical ordered sets, Order 1 (1984) 3–5.

[9] T. Hiraguchi, On the dimension of partially ordered sets, Sci. Rep. Kanazawa Univ. 1, 77-94, MR17, 19 (1951).

[10] D. Kelley, The 3-irreducible partially ordered sets, Canad. J. Math. 29 (1977) 367–383.

[11] D. Kelley and W.T. Trotter, Jr., Dimension theory for ordered sets, in: I. Rival, ed., Ordered Sets (Reidel, Dordrecht, 1982) 171–211.

[12] M. Pouzet and I. Rival, Structural arithmetic of the extension lattice of an order, preprint, 1991.
[13] I. Rabinovitch and I. Rival, The rank of a distributive lattice, Discrete Math. 25 (1979) 275–279.
[14] I. Rival, Optimal linear extensions by interchanging chains, Proc. Amer. Math. Soc. 89 (1983) 387–394.
[15] I. Rival and N. Zaguia, Constructing greedy linear extensions by interchanging chains, Order 3 (1986) 107–121.
[16] M.M. Syslo, Remarks on Dilworth partially ordered sets, in: H. Noltemeir, ed., Proceedings of the WG'85 (Trauner Verlag, Linz, 1985) 355–362.
[17] E. Szpilrajn, Sur l'extension de l'ordre partiel. Fundam. Math. 16 (1930) 386–389.
[18] W.T. Trotter, Jr., Dimension of the crown $S_n^k$, Discrete Math. 8 (1974) 85–103.
[19] W.T. Trotter and J.I. Moore, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, Discrete Math. 15 (1976) 71–82.
[20] E.S. Wolk, A note on the comparability graph of a tree, Proc. Amer. Math. Soc. 16 (1965) 17–20.